Chapter IV:

Introduction to General Equilibrium With Production

> Under the assumed conditions--(a) two commodities, (b) produced by two factors of production, and (c) where trade leaves something of both commodities produced but at a new margin--it has been unequivocally demonstrated that the scarce factor must be harmed absolutely [by a reduction in protection]. (Stolper and Samuelson, 1941, pg. 350)

In this chapter we introduce some basic techniques and results from general equilibrium theory. We will be particularly interested in the simple two-factor, two-good (2×2) model commonly used in international trade theory and a related model with three factors of production and two goods where two of the factors are sector-specific and one is intersectorally mobile. Before pursuing the details of this analysis it may prove useful first to provide some notion of what is involved in general equilibrium modeling and then to provide some motivation for the use of general equilibrium theory.

General equilibrium analysis deals explicitly with the interdependence among households and firms as mediated by markets. The basic units in the analysis are: *households*, which are represented by their portfolio of factors (e.g. capital, labor, land, etc.) and their preferences for final consumption goods; and *firms*, which are represented by the outputs they produce and their technologies, from which we can determine their demand for factors of production. Households and firms are linked via: *factor markets* where households rent the services of their factors of production to firms, generating income for the households; and *commodity markets*, where firms sell final consumption goods to consumers, generating income for firms. These markets may or may not be competitive, but they are interdependent. That is, decisions in one market have an effect on all the other markets. General equilibrium theory seeks to study this system as a whole. By focusing on the interdependence of markets, we are able to clearly identify conflicts of interest based on the economic attributes of various agents: Producers v. Consumers; Capital Owning v. Labor Owning Households; Industry v. Industry; and with more detailed models we can produce more complex patterns of conflict (e.g. Inter-Generational Conflict, Inter-Gender Conflict, International Conflict, etc.).

So why adopt a general equilibrium strategy to representing the economy? There are a number of quite vital political-economy traditions that focus their modeling efforts primarily on partial equilibrium representations of the economy (e.g. the Chicago school). For many types of problems this is a sensible strategy. For example, many local regulatory problems can be treated as essentially partial equilibrium in nature and the obvious focus on political interaction between producers and consumers makes good sense. However, for a wide range of issues in the political-economy of trade policy a general equilibrium approach would seem to be essential. Perhaps most importantly, partial equilibrium analysis seems to imply that the most significant axis of political competition over a policy is between producers and consumers. At least with respect to final consumers, however, this is a political battle that is rarely significant to trade policy outcomes. While it is surely the case that political interaction between producers and consumers of intermediate products plays a significant role in the political-economy of trade policy, general equilibrium analysis provides a unified framework that can incorporate a wide array of possible axes of political competition. In addition, the general equilibrium framework allows us to take full advantage of the large number of results in the main body of trade theoretical and empirical research and, in turn, to link our results back to that body.

In the development of our analysis we will follow the usual approach of trade economists (and most other applied economists) and begin with the *minimally complex* model necessary for our analysis. It is an old joke that "reality is its own and only complete model". Thus, all analysis is based on heroic simplification, and choosing the degree of simplification is an essential element of any theoretical or empirical effort. Given the extreme distance of any practical model from the "only complete model", there is much to be said for the strategy of starting with the minimally complete model. That is, we start with the simplest possible model that incorporates the key elements with which we are concerned. Although we will develop the full set of assumptions in the next section, it should be clear that a model of international trade requires at least 2 goods (otherwise there would be nothing to exchange), while the explanation of trade in terms of varying factor-endowments which has proved extremely useful in the development of trade theory requires the existence of at least two factor of production. As we shall find in chapters 9-11, the existence of multiple factors of production will also prove to be essential to our political economic analysis.

While minimalist models are extremely useful for developing the logic of a particular channel of causation in economic or political-economic analysis, the interpretation of data or particular events in terms of such models requires caution. A good example is the use of standard econometric methods to "test" the predictions of the 2×2 model. Leaving aside the history of problematic inference in the history of the literature on tests of endowment-based trade models, there is the issue of our priors on the empirical status of correctly phrased hypotheses: given the utter simplicity of the model, what kind of empirical result would convince us that the model is capturing something important in the explanation of international trade patterns.[1] Perhaps even more problematic is the construction of a specific model to rationalize a particular event of set of "stylized facts". Contrary to the doubtful methodological strictures of Friedman's well-known essay, since we can construct models to generate virtually any outcome, an essential element in the evaluation of a body of theory relates to the reasonableness, broadly construed, of the assumption structure. With respect to explaining the pattern of trade among countries and the distributional implications of that trade, the *reasonableness* of the claim that endowment differences among countries and among households matters seems undeniable.

Since we can rely on the detailed treatment of consumer choice in chapter 2, we begin with only a brief review of the assumptions on households for the reader's convenience. From there we proceed to a more detailed analysis of the supply-side of the model and a characterization of general equilibrium in such models.

Key Behavioral and Institutional Assumptions

We begin with the assumptions that characterize the institutional environment of our simple economy. The essential behavioral assumption that sets the entire system in motion is the assumption that all agents (i.e. households and firms) are characterized by **rationality** (as described in chapter 2).

Complete and Perfect Markets: Throughout most of the second part of this monograph, we will assume that all markets (for goods and factors of production) exist and are perfectly competitive. Thus,

[1]See Leamer and Levinsohn (1995) for a detailed discussion of these issues in the context of a survey of modern empirical research on the predictions of trade theory. An extraordinarily good treatment of the construction of a research program that takes explicitly account of these issues is Leamer (1984).

we are assuming that there are: complete, perfect and fully alienable property rights; the existence of a market for all goods and all factors of production; and perfect competition (including full information, and costless and instantaneous arbitrage) in all of those markets. We will use this assumption to ensure that, in equilibrium, all markets clear (in particular, there will be full-employment of factors of production). **Uniform Quality within Categories**: this means that all goods of a given type (say automobiles) are of the same quality and all factors of production (say labor) are of the same quality. This last assumption, along with rationality and our institutional assumptions on markets will ensure that, in equilibrium, all factors receive the same wage and all goods are transacted at the same price. We will assume that the national economy is **economically small** in the international economy. That is, the country is assumed to be a price-taker in world markets. This simplifies the early development of the model and will be relaxed in chapters 13-15. Finally, we have already noted that we will assume that there are **two goods and two (or three) factors of production**. We'll call this an institutional assumption. Again, the idea here is to keep our dimensionality as low as possible consistent with our goal of building a model of international trade. If we are going to trade, we'll obviously need at least two goods. In chapter 5 we examine the consequences of relaxing the dimensionality assumption.

Assumptions About Demand

As in chapter 2, we assume that individual preferences are such that they can be represented by real-valued utility functions defined on bundles of consumption goods. If we let $\boldsymbol{x}_h$ be a particular bundle of consumption goods, then household utility is given by:[2]

$$:_h = u^h(\boldsymbol{x}_h).$$

Furthermore, as developed in chapter 2, using the assumption that households optimize subject to a budget constraint, we can represent preferences with the indirect utility function

$$:_h = v(\boldsymbol{p}, (^h),$$

where $(^h$ is household income (derived from renting its endowment at the market rates) and $\boldsymbol{p}$ is the vector of commodity prices.

For much of our analysis our exposition will be dramatically simplified if we assume that we can represent community tastes with a *community utility function* that inherits all the properties of the individual utility functions. Since, in our political-economic analysis we will want to distinguish the differential effects of policy on different households, the easiest way to accomplish this is to assume that *all households' preferences are identical and homothetic*. In this case we can use community utility functions to characterize the economic equilibrium:

$$: = U(\boldsymbol{x}),\ x_j = 3 x_j^h\ ,\ œ\ j\ 0\ J, \qquad \textbf{(1)}$$

where J is the set of all available goods (in our simple case, $J = \{1,2\}$). As in the case of individual choice, we can represent these community preferences by the level sets of the function projected into commodity-space. Our assumptions insure that these **community indifference curves** are: *Negatively sloped* (reflecting our assumption that goods 1 and 2 are "goods", that is, giving up some of one good requires an increase in the endowment of the other good to remain at the same level of utility;

[2]Since we are assuming that there are only two consumption goods $\boldsymbol{x}_h = \{x_1^h, x_2^h\}$

non-intersecting (reflecting our assumption that collective preferences are transitive); and *bowed in toward the origin* (this shape reflects our assumption that there are diminishing marginal rates of substitution--ie. that the marginal utility of consuming one good falls relatively to the other as the quantity consumed of the first good is increased and of the second decreased). The slope at any point on an indifference curve gives the ratio of marginal utilities in consumption (called the **marginal rate of substitution, MRS**) at that point. That is, the slope is equal to the ratio of marginal utilities, which is the marginal rate of substitution. With homotheticity, this slope is constant along any specific ray from the origin.

Now suppose that the community's income is equal to ' and we want to know what is the best allocation of ' between goods 1 and 2 if commodity prices are fixed at p_1 and p_2 by the small country assumption. This problem can be written as:

$$\max_{x_1, x_2} m = U(x_1, x_2)$$
$$\text{s.t. } \Gamma = p_1 x_1 + p_2 x_2$$

For use in graphical illustration, it is useful to write the national income constraint in slope-intercept form as:

$$x_2 = -\frac{p_1}{p_2} x_1 + \frac{\Gamma}{p_2}.$$

Note that the slope of the line is the commodity-price ratio. It should be clear just from looking at the diagram that, given the shape of the indifference curves, the highest possible level of national welfare is available where a community indifference curve is just tangent to the national income constraint. This analysis implies that, in equilibrium, the ratio of marginal utilities is equal to the price ratio it observes in the (world) market. This makes perfectly good economic sense. Suppose that a representative household is holding some other bundle on its budget constraint. We know that one, and only one, indifference curve passes through that point. We also know that, since it is not a tangency, it must cut the budget constraint. Suppose that the slope of the indifference curve (i.e. the MRS) is greater than the slope of the budget constraint (ie. the price ratio). This means that:

$$MRS = \frac{\partial u(\bullet)/\partial x_1}{\partial u(\bullet)/\partial x_2} > \frac{p_1}{p_2} = P.$$

But that means that the satisfaction that this household gets from good 1 relative to good 2 is greater than the relative price of good 1. Thus, this household should be willing to trade some of its good 2 for some additional good 1 at the market price. Since, by assumption, all households make this arbitrage, in equilibrium the community will be at a point of tangency. At that point, the community's internal tradeoff between 1 and 2 (given by the MRS) is equal to the market tradeoff (given by the relative price). Our assumption of homotheticity guarantees that, as long as relative prices stay the same, consumers will choose to consume the goods (1 & 2) in the same proportion. Thus, the points at which a ray from the origin (called an **income-consumption line** in this context) cuts each indifference curve all have the same slope (which equals the MRS at that point). This will make the graphical presentation of our analysis considerably easier.

Assumptions About Production

Recall from our institutional assumptions that the economy is endowed with fixed quantities of two factors of production that, for concreteness, we will refer to as capital (*K*) and labor (*L*). We will denote this endowment: $\overline{\mathbf{z}} = \{\overline{K}, \overline{L}\}$. Each of the two productive sectors has technologies that can be characterized by **neoclassical production functions**:

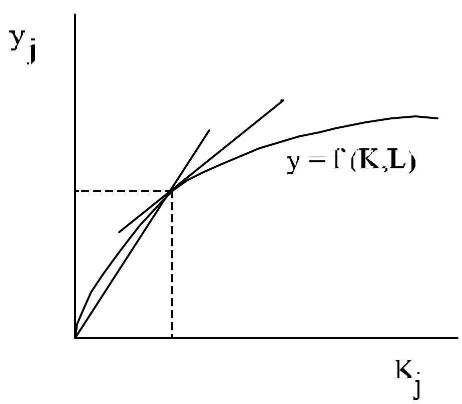
$$y_j = f^j(K_j, L_j)\ .$$

We will assume that: both factors are essential to production

$$f^j(K,0) = f^j(0,L) = 0\ ;$$

the production functions are linear homogeneous, twice differentiable, and strictly quasi-concave. Linear homogeneity is just a fancy way of saying that production occurs under conditions of **constant returns to scale**.[3] The assumption of strict quasi-concavity says that there are **positive, but diminishing, returns to increasing one of the factors while holding the other constant**.[4] That is,

$$\frac{\partial y_j}{\partial K_j} > 0;\ \frac{\partial^2 y_j}{\partial K_j^2} < 0;\ \frac{\partial y_j}{\partial L_j} > 0;\ \frac{\partial^2 y_j}{\partial L_j^2} < 0.$$



One convenient graphical representation of the production function holds one input, say *K*, constant at some positive level (recall that there is no output without positive input of each factor) and shows the effect of varying the input of the other factor (*L*) on the firm's output. The above diagram illustrates our assumptions: the graph

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[3]Linear homogeneity: "*f*(**x**) = *f*("**x**).

[4]A function *f*:SdR^{n}6R, for S convex, is strictly quasi-concave if the upper contour set U(f,b) is convex for all b0 R.

of the production function passes through the origin--i.e. zero output with zero input of K; the positive slope shows that MPP_{Kj} is always positive, while the decreasing slope shows diminishing marginal product; the slope of a ray from the origin through the actual production point shows the **average product** of capital:

$$\frac{y_j}{K_j} = a_{Kj}.$$

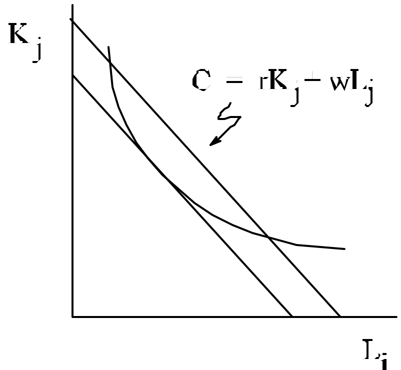
For future use, we will denote the inverse of average product, called the *input-output coefficient*:

$$a_{KJ} = \frac{K_j}{y_j} = \frac{1}{a_{Kj}}.$$

This gives the input of capital in one unit of output of good j. The input-output coefficient for L in sector j is defined in the same way.

The other standard graphical representation of technology shows all sets of factor-inputs that yield the same levels of outputs and are, thus, called *isoquants*.[5] Our assumptions guarantee that the isoquants are "bowed-in" to the origin of K-L space. The slope of an isoquant at any point gives the ratio of marginal physical products (MPP_K and MPP_L) at that point (called the Marginal Rate of Technical Substitution, MRTS):

[5]One way of thinking about isoquants is that they show the outline of slices taken out of the three-dimensional graph of the production function at a fixed height (i.e. level of output)--i.e. the *level-set* of the production function for a given output. This can be compared to the previous graph which showed a vertical slice, taken at a fixed level of K input.



$$y_j = f^j\left(K_j, L_j\right);$$
$$dy = \frac{\partial f^j(\bullet)}{\partial K}dK + \frac{\partial f^j(\bullet)}{\partial L}dL$$
$$dy = \left(MPP_K\right)dK + \left(MPP_L\right)dL$$

Since every point on an isoquant represents the same level of output, along an isoquant dy = 0, therefor

$$\frac{dK}{dL} = -\frac{\partial f^j(\bullet)/\partial L}{\partial f^j(\bullet)/\partial K} = -\frac{MPP_L}{MPP_K} \equiv MRTS_{LK}.$$

Our assumptions of perfect competition, factor mobility and rationality insure that factors will be paid their value of marginal product ($VMP_j = p_j\, MPP_j$):

$$w = VMP_L = p\left(MPP_L\right) = p\left(\frac{\partial y}{\partial L}\right);$$
$$r = VMP_K = p\left(MPP_K\right) = p\left(\frac{\partial y}{\partial K}\right).$$

Thus, in equilibrium, the slope of the isoquant is equal to the *w/r* ratio. Suppose that firm in sector 1 seeks to produce some level of output, say $\bar{y}$.

From the rationality assumption, we know that the firm will seek to minimize the cost of producing that output. If the firm faces fixed factor prices (*w* and *r*), we can write its problem as:

$$\min_{K_1, L_1} c_1 = wL_1 + rK_1$$
$$\text{s.t.}\, f^1\left(K_1, L_1\right) = \bar{y}$$

But we can rewrite the expression for cost in the slope-intercept form:

$$K_1 = -\frac{w}{r} L_1 + \frac{c_1}{r}.$$

This should be recognizable as the equation of a straight line in *K-L* space with a slope of *w/r* (i.e. the ratio of unit factor costs). Any point on the $\bar{y}$ isoquant yields a level of output of good 1 equal to $\bar{y}$. But costs are minimized only at a tangency between the $\bar{y}$ isoquant and the isocost line. Suppose instead that the firm hires a combination of *K* and *L* such that the isoquant cuts the isocost from above, i.e.:

$$MRTS_{LK}^{1} = \frac{\partial f^1(\bullet)/\partial L_1}{\partial f^1(\bullet)/\partial K_1} > \frac{w}{r}$$

But this means that the relative productivity of *L* exceeds the relative cost of *L*. Thus the entrepreneur will attempt to substitute *L* for *K* along the isoquant. It is easy to see that the new point is on a lower isocost. This process will continue until it is no longer profitable--ie. until a point of tangency is reached.[6] At which point: cost is minimized; and MRTS = *w/r*. Furthermore, our assumptions ensure that the isoquant is smoothly bowed-in to the origin, so *there is a 1-to-1 relationship between the w/r ratio and the K/L ratio in production*, and our assumption of constant returns to scale means that this relationship is independent of scale of production.

Our assumption of constant returns to scale, along with our institutional and behavioral assumptions, buys us one additional useful fact: **zero economic profits**. That is, all proceeds from the

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[6]This is just the arbitrage argument that we have already seen in characterizing the process by which households pick out optima using local information about price and preference.

production and sale of goods are distributed to factors of production. We have already argued that factor mobility and perfect competition in all markets guarantees that factors will be paid the value of their marginal products. Now we use a property of linear homogeneous production functions called Euler's Law, to show that this completely exhausts the returns from production. Constant returns to scale means that:

$$f^j\left(l K_j, l L_j\right) = l f^j\left(K_j, L_j\right).$$

Since this identity holds for all values of *K* and *L*, we can differentiate both sides with respect to 8, using the chain rule:

$$\frac{\partial f^j(\bullet)}{\partial\left(l K_j\right)}\frac{\partial\left(l K_j\right)}{\partial l} + \frac{\partial f^j(\bullet)}{\partial\left(l L_j\right)}\frac{\partial\left(l L_j\right)}{\partial l} = \frac{\partial\left(l y_j\right)}{\partial g}$$

Noting that: $\mathrm{M}(8L)/\mathrm{M}8 = L$ (similarly for *K*); and that, since this is an identity that holds for all 8 and all (*K,L*) then it must hold for 8=1. Thus:

$$f_K^j K_j + f_L^j L_j = f^j.$$

Multiplying both sides by p_1 and recalling the definition of VMP gives

$$\left(pf_K^j\right) K_j + \left(pf_L^j\right) L_j = rK_j + wL_j = pf^j.$$

Note that if we divide this last equation through by y_j, so that we are dealing with per unit costs, we get the following **zero profit equilibrium condition**:

$$p_j = a_{Lj} w + a_{Kj} r,\ j \in \{1,2\}. \qquad \textbf{(2)}$$

A ray from the origin is called an **expansion path**. Since the slope of the expansion path gives the *K/L* ratio in production, our assumptions about the production functions guarantee that any given expansion path cuts all members of the family of isoquants at points with the same slope. Therefor, the *w/r* ratio determines the equilibrium *K/L* ratio; and constant returns to scale means that, defining any arbitrary amount of output as a "unit", we can find all other output levels as simple multiples of that unit.

A final technological assumption, that will play a significant role in much of our later analysis, is the **factor-intensity condition**: We will assume that one sector, say 1, can be characterized as *K*-intensive relative to the other at any given relative factor prices. That is, for any relevant *w/r* ratio, it will always be the case that

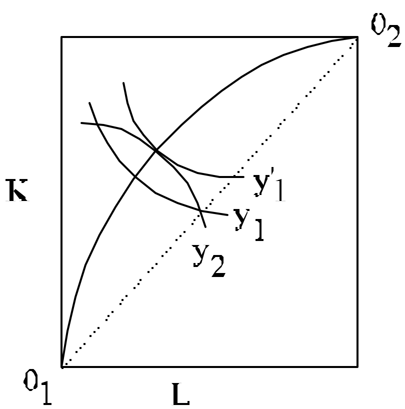
$$\frac{K_1}{L_1} > \frac{K_2}{L_2} \Leftrightarrow \frac{a_{K1}}{a_{L1}} > \frac{a_{K1}}{a_{L1}}. \qquad \textbf{(3)}$$

The second inequality follows from constant-returns to scale $\left(\frac{a_{Kj} y_j}{a_{Lj} y_j} = \frac{K_j}{L_j} \ \forall y_j \text{ and } \frac{w}{r} \text{ fixed} \right)$. Note, in particular, that we are assuming that there are *no factor-intensity reversals*. This is a far from innocent assumption for which there is no particular empirical support. Nonetheless, it is quite standard in trade theory and we will use it extensively in the sense that many of our results will rely on the assumption holding. Using the isoquant representation of technology, these assumptions mean that, for any common factor price ratio, the expansion path of the *K*-intensive sector will always lie above the expansion path of the L-intensive sector.

Production Possibilities and National Income

Now that we have established the properties of technologies in the individual industries, the next step is to identify the set of technologically efficient outcomes.[7] That is, we want to be able to represent all the combinations of outputs available to the country given its technology (represented by the production functions) and its factor endowments (i.e. the inputs to production). One such representation is the *production-box diagram*.

Construct a box whose height is given by the economy's endowment of capital, and whose width is given by its endowment of labor. Every point in the box represents an allocation of the economy's resources between the two industries. Perfect competition and costless inter-industry mobility ensures **full employment**:



$$\begin{aligned}\bar{L} &= L_1 + L_2 = a_{L1}y_1 + a_{L2}y_2 \\ \bar{K} &= K_1 + K_2 = a_{K1}y_1 + a_{K2}y_2\end{aligned} \qquad \textbf{(4)}$$

Now we want to introduce our technologies into the diagram. Using the SW corner of the box as an origin, project the isoquant map for good 1 into the box. Using the NE corner of the box as an origin, project the isoquant map for good 2 into the box. All efficient combinations will lie on the locus of tangencies between good 1 and good 2 isoquants. From any point of non-tangency we can always find an alternative allocation of resources between the two industries

[7]It is probably worth noting that our assumption of constant returns to scale, along with our behavioral and institutional assumptions, means that firm size is indeterminate. However, it also means that there is no analytical loss in dealing with the industry as the unit of analysis, since our assumptions ensure that the industry production function inherits all the properties of the firm production functions.

that permits, at least, increased output of one good without decreased ouput of the other good. Since we have already seen that the slope of an isoquant is equal to the MRTS, the tangency result implies that MRTS ratios are equalized between sectors. This makes sense given our assumptions of perfect competition and costless mobility. We know that *w* and *r* will be equalized, but we also know that competitive firms will pay factors their VMP_L and VMP_K. But this implies that $MRTS_1 = MRTS_2$.

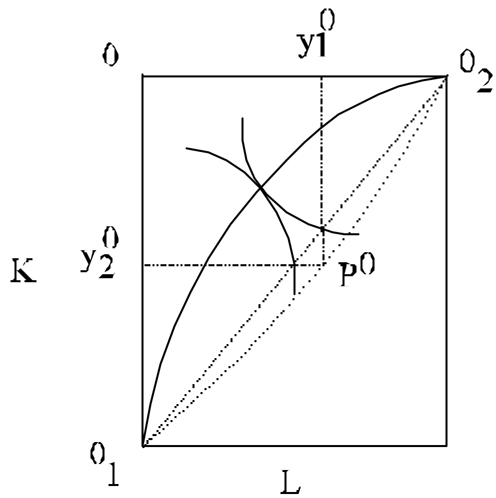
$$MRTS^1 \equiv \frac{MPP_L^1}{MPP_K^1} = \frac{P_1\left(MPP_L^1\right)}{P_1\left(MPP_K^1\right)} = \frac{VMP_L^1}{VMP_K^1} = \frac{w}{r} = \frac{VMP_L^2}{VMP_K^2} = \frac{P_2\left(MPP_L^2\right)}{P_2\left(MPP_K^2\right)} = \frac{MPP_L^2}{MPP_K^2} \equiv MRTS^2$$

Note, by the factor-intensity assumption, that since good 1 is always *K*-intensive relative to good 2, the locus of tangencies lies everywhere above the diagonal linking the two origins.

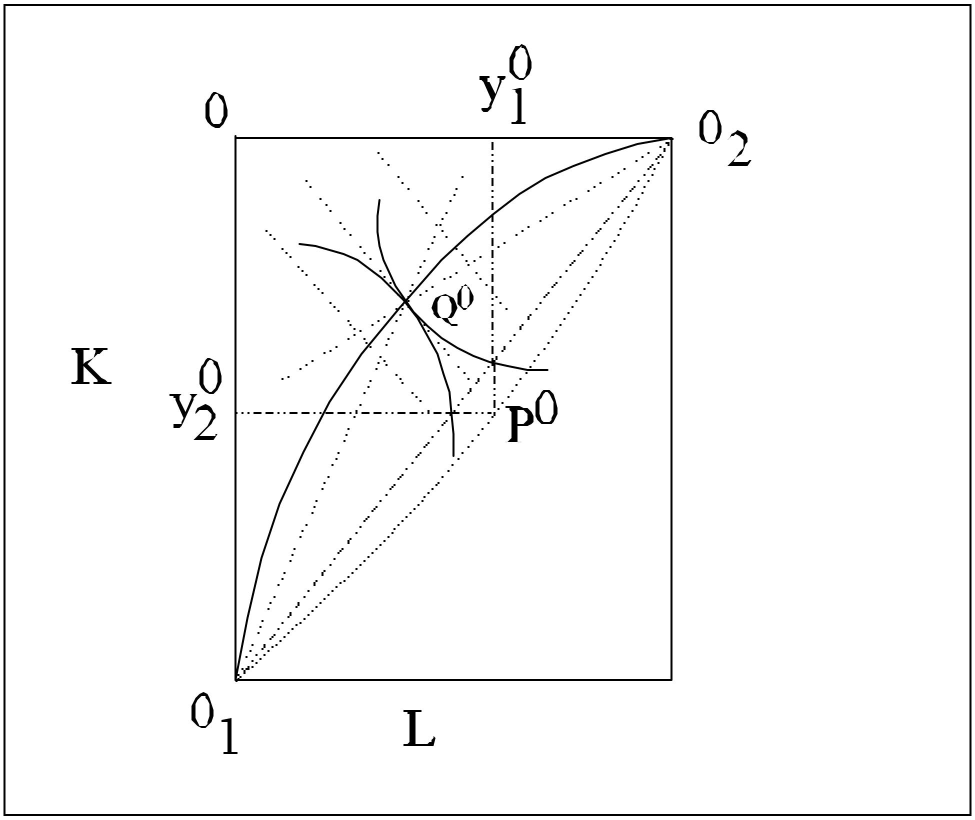
For many problems it is more convenient to conduct our analysis in output-space, rather than in the input-space used by the production box. The diagram showing the economy's production opportunities is called the *production set* (whose boundary is the *efficient frontier* or *production possibilities curve*), is easily derived from the production box diagram. Given rationality, perfect competition, and perfect factor mobility, we need only focus on efficient production combinations. We already know that the locus of tangencies between isoquants in the production box gives all the efficient output combinations. All we need is a way of getting from the locus of tangencies to the PPC. For illustrative purposes we will adopt an approach called the Savosnik technique (Savosnik, 1958).[8]

First, we note some preliminary facts: with constant returns to scale, the output level associated

[8]The treatment here follows that of Johnson (1971).



with any isoquant can be measured by its distance from the origin along any ray from the origin; the diagonal 0_1-0_2 is a common ray for the two isoquant systems; and 0_1 represents zero output of good 1 and 0_2 represents zero output of good 2. Now we let the NW corner be the origin for the PPC, and let 0_1 represent maximum production of good 2 and 0_2 maximum production of good 1 on the PPC. For any point on the locus of tangencies, say Q^0, to find output y_1^0: find the intersection of the relevant isoquant with the diagonal; drop a perpendicular to the axis measuring y_1 (i.e. 00_2). Do the same for y_2^0. These points are the coordinates on the PPC corresponding to q^0 on the locus of tangencies. Call this point P^0. This can be repeated for every other point on the locus of tangencies to trace out the PPC.



Now that we have developed a simple graphical approach to production for general equilibrium, we pursue a few useful properties production in the neoclassical model in a bit more detail. First, *the shape of the production possibilities curve.* Given the assumptions we have just made, we can show that the PPC is smoothly "bowed outward" from the origin, reflecting *increasing opportunity costs of transforming good 1 into good 2*. P^0 must be "above" the diagonal (referred to the orign of the PPC), but this is not enough to prove that the PPC is strictly concave to the origin. We need a couple more steps. Assume factor prices are temporarily held constant and consider equal increases and decreases in y_1 about point Q^0, to isoquants y_1^{0+1} and y_1^{0-1}. With constant factor prices, these shifts would involve equal decreases and increases in y_2, to isoquants y_2^{0-1} and y_2^{0+1} respectively. But constant factor prices would entail an excess demand for K at (y_1^{0+1}, y_2^{0-1}) and an excess supply of K at (y_1^{0-1}, y_2^{0+1}). Note: what we have done graphically is shift the common tangent and identify the (K,L) combinations consistent with tangencies on the original expansion paths. The fact that the two new production points are not tangent is insured by the fact that the expansion paths do not intersect at these points. We already know that equilibrium entails a tangency between y_1^{0+1} and an y_2 isoquant when y_1 increases and between y_1^{0-1} and an y_2 isoquant when y_1 decreases. If we retain the changes in good 1, each equilibrium good-2 isoquant must involve a lower level of output of good 2 than y_2^{0-1} and y_2^{0+1} respectively. Thus, an increase in good 1 form y_1^{0-1} to y_1^{0} must involve a smaller reduction in good 2 output (from something less than y_2^{0+1} to y_2^{0}) than an increase in good 1 from y_1^{0} to y_1^{0+1} (from y_2^{0} to something more than y_2^{0-1}). Since this proof holds regardless of the magnitude of the equal changes in the output of good 1, the PPC must be strictly concave to the origin.

Suppose that relative prices are exogenously given to the country (this assumption is called the "small country assumption" because the country is a price taker in world markets). We want to find the levels of output of the two goods that maximizes national income. We can take any point on the production frontier, (y_1, y_2), and use the given prices to find the value of national income implied by that output, ', as:

$$' = p_1 y_1 + p_2 y_2 . \quad \textbf{(5)}$$

This is easily transformed into slope-intercept form to give

$$y_2 = -\frac{p_1}{p_2} y_1 - \frac{\Gamma}{p_2}.$$

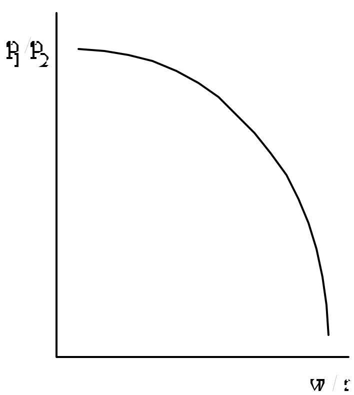
There are two important things about this line: Its slope gives the relative price ratio; and its distance from the origin shows the level of national income. Note that every point on a given line shows combinations of y_1 and y_2 that yield the same level of income. Also note that, for given prices, every output combination lies on one, and only one, line. Now suppose that, at the given output combination (y_1,y_2), the price line cuts the production from above, i.e.

$$P := \frac{p_1}{p_2} > \frac{\partial y_2}{\partial y_1} = \frac{1}{l^*} = MRT.$$

This means that the market price of good 1 (i.e. p_1 relative to p_2) is greater than the opportunity cost of good 1 (ie. the MRT). This situation will induce entrepreneurs to shift out of production of good 2 (() y_2 < 0), into production of good 1 (() y_1 > 0) until the inequality no longer holds. Note that each such marginal shift increases the value of national income. This is represented graphically by successive outward shifts in the line representing the value of national income. When the inequality no longer holds, it must be the case that the relative price is equal to the MRT (ie. the price line is tangent to the PPF). Entrepreneurs no longer have an economic incentive to shift output between sectors. There are no national income lines consistent with feasible production that represent a higher level of national income.

There is a 1-to-1 relationship between relative commodity prices and relative outputs. We have just shown that the efficient production frontier is strictly concave. This means that every point on the production frontier has a unique slope. We have also seen that the value of national product is maximized where the national income line, whose slope is determined by relative prices, is tangent to the efficient frontier. Thus, at an interior equilibrium, every relative price picks out a unique output combination. We would have expected this since given our production and competition assumptions, we know that $p_j = c_j$. This requires that the ratio of marginal costs be equal to the commodity price ratio. Given the previous demonstration of smoothly increasing opportunity costs, this implies that there is a unique commodity price ratio for every non-specialized efficient output combination.

There is a 1-to-1 relationship between relative commodity prices and relative factor prices. From the production box diagram, we know that every efficient production point is characterized by a specific *w/r* ratio. This is given by the slope of the common tangent at the relevant point on the contract curve. This reflects our expectation that industries cannot be paying different *w* and *r* in an efficient equilibrium. But every point on the contract curve relates to one-and-only-one point on the PPC. Since every point on the PPC represents a unique p_1/p_2, and every point on the contract curve gives a unique *w/r*; then there is a 1-to-1 relationship between p_1/p_2 and *w/r*. Note that this is a purely technological relationship between MRTs and MRTSs, it applies to all countries regardless of size or relative endowment (subject only to the constraint that both goods are being produced). We can plot this relationship on a simple graph. Since both countries possess the same technological opportunities, they can both be shown on the same graph.



A Fundamental Result for Political Economy: The Stolper-Samuelson Theorem

One of the real advantages of this model from the perspective of political-economy modeling is that it allows us to explicitly consider the effects of changes in the economic environment on the income distribution. The key result here is called the **Stolper-Samuelson Theorem**, which states that an increase in the relative price of a good raises the return to the factor used intensively in the production of that good relative to all other prices and lowers the return of the other factor relative to all other factors. The value of this result to political-economic analysis should be obvious: by linking policy-induced changes in price to the real returns to household factor-portfolios, the task of deriving household preferences over policies is rendered considerably easier. This is precisely what we will do in chapter 8.

The Stolper-Samuelson theorem really has two parts: the first part links commodity prices to factor-prices in relations of *friends and enemies*; the second relates to *magnification*. We will say that a commodity and a factor are "friends" if an increase in the commodity-price leads to an increase in factor-price; and we will call them "enemies" if an increase in the commodity price leads to a decrease.[9] Magnification refers to the factor-price changes that are proportionally greater (i.e. magnified) than the commodity-price changes. The Stolper-Samuelson theorem asserts that, in an HOS economy: each commodity is a friend to one factor and an enemy to the other; we can unambiguously identify friends and enemies by factor-intensity; and these relations are magnified. We can easily illustrate the friends and enemies part of the theorem using the same logic that we used to show that the efficient production frontier is bowed-out from the origin. Suppose that the price of 1 rises while the price of 2 remains

[9]The "friends and enemies" language comes from Jones and Schienkman (1977), but our use is slightly different. In Jones and Schienkman the friend and enemy relations require positive and negative changes in *real* factor returns, while we want to emphasize magnification as a distinct, and distinctly useful, property.

unchanged. Entrepreneurs in sector 1 will begin to bid factors away from production in sector 2. But we have already seen that this creates an excess demand for *K* (the factor used intensively in 1 production) and an excess supply of *L*. This causes *r* to rise and *w* to fall. The same argument applies, *mutatis mutandis*, for the price of good 2. This establishes the friends and enemies part of the result.

The logic of the magnification effect is relatively straightforward. As the appendix shows, total differentiation of the zero profit conditions (i.e. equations (2)) and manipulation shows that, in proportional changes (represented with "hats"), a price change must be equal to a distributive share-weighted average of factor price changes. That is, letting 2_{ij} be the share of factor *i* in the value of output *j*, i.e. $q_{ij} = \frac{a_{ij} w_i}{p_j}$:

$$\begin{aligned} \hat{p}_1 &= q_{L1}\hat{w} + q_{K1}\hat{r} \\ \hat{p}_2 &= q_{L2}\hat{w} + q_{K2}\hat{r} \end{aligned} \qquad \textbf{(6)}$$

This weighted average property means that each price change must be weakly bounded by the factor-

$$\hat{r} > \hat{p}_1 > \hat{p}_2 > \hat{w}.$$

price changes (i.e. the proportional changes could be equal). Thus, the two commodity-price changes must fall weakly between the two factor-price changes. However, our factor-intensity assumption ensures that, since the commodity-prices change differentially, the chain involves strong inequalities. That is:

This is the Stolper-Samuelson theorem. Note that this applies to any change in relative prices (i.e. p_2 need not remain constant, it only need rise by less than p_2). Given the centrality of this result, appendix

1 to this chapter develops a formal proof in more detail.

Using the tools we have developed so far, illustrating the magnification effect is a bit tricky. This is because, while the Stolper-Samuelson theorem is about a relationship between prices, all of our graphical apparatus to this point have been defined in quantity spaces. However, there is an alternative representation of technology that permits an extremely easy demonstration: the *unit cost curves*. The unit cost curve shows all *w-r* combinations that, given a sector's technology, solve the problem of minimizing the cost of producing one unit of output. That is

$$c^j(w,r) = \min_{K_j,L_j}\left\{wL_j + rK_j \,\middle|\, f^j(K_j,L_j) \geq 1, K_j \geq 0, L_j \geq 0\right\}.$$

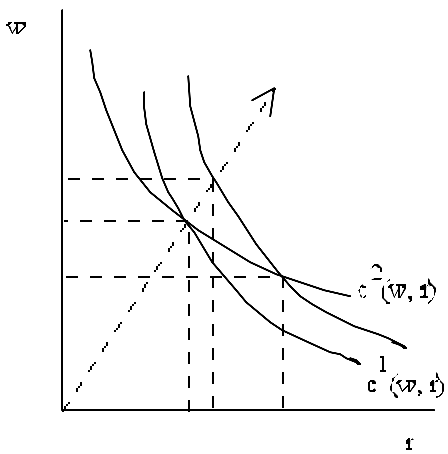
The unit cost function has several useful properties. Because we have assumed constant returns to scale, the unit cost curve fully characterizes the technology in input-price space in exactly the way that an isoquant fully characterizes the technology in input-quantity space.[10] It is easy to see that the $c^j(w,r)$ will be linear homogeneous in $\{w,r\}$. That is, if both w and r increase by 8 percent, the w/r ratio is unchanged so the equilibrium inputs will be unchanged, and the cost of producing one unit must also increase by 8 percent. Since the cost function is concave in $\{w,r\}$, the level set (in $\{w,r\}$-space) is convex. Finally, an extremely useful result called *Shephard's lemma* states that the partial derivatives of the unit cost function give the factor-input demands for a unit of output:

$$\frac{\partial c^j(w,r)}{\partial w} = a_{Lj} \text{ and } \frac{\partial c^j(w,r)}{\partial r} = a_{Kj}.$$

[10]In fact, the unit-value isoquant and the unit cost curve are dual to one another. See Darrough and Southey (1977) for a very useful graphical treatment of this relationship.

Thus, the gradient vector of $c^j(w,r)$ is the optimal input vector for a unit of output, so $\{a_{Lj}', a_{Kj}'\}$ is orthogonal to the plane supporting $c^j(w,r)$ at $\{w', r'\}$. Another way of saying the same thing is that the slope of the unit cost curve in *w-r* space gives the equilibrium *K/L* ratio.

Suppose that good 1 is factor *K*-intensive, relative to good 2, so that the unit cost curve for good 1 always cuts the curve for good 2 from above. We have already seen that the gradient vector at any point on the unit cost curve gives the $\{a_{Lj}, a_{Kj}\}$-vector. Thus, the vectors at the intersection which determines the equilibrium $\{w,r\}$-vector, illustrates the *K*-intensity of sector 1 we have assumed. The friends and enemies part of the Stolper-Samuelson theorem is now easily illustrated: suppose that p_1 rises while p_2 is unchanged. The unit cost curve for sector 1 shifts outward, while the other retains its position. At the new intersection, w_1 has increased and w_2 has fallen. This establishes:



$$\hat{p}_1 > \hat{p}_2 = 0 > \hat{w}_2.$$

The magnification effect is only marginally more difficult to see. Construct a ray from the origin through the original equilibrium $\{w,r\}$-vector. Find the proportional increase in w where the ray cuts the new unit cost curve for sector 1. It is easy to see that the equilibrium w_1 exceeds the proportionally increased w_1. Thus

$$\hat{r} > \hat{p}_1 > \hat{p}_2 > \hat{w}.$$

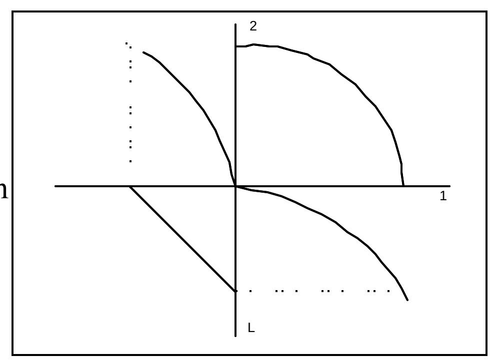
The Ricardo-Viner ("Specific-Factors") Model

An alternative to the 2-factor × 2-good, Heckscher-Ohlin-Samuelson (HOS), model that has been extensively used in political-economy modeling is the 3-factor × 2-good, specific-factor (or Ricardo-Viner), model. As with the HOS model, we continue to make all of our behavioral and institutional assumptions, except we now assume that there are **three factors of production**: intersectorally mobile labor and two types of sector-specific capital.[11] Since we continue to assume perfectly competitive factor markets, there will be full employment of all factors:

$$\begin{aligned} L_1 + L_2 &= \overline{L} \\ K_1 &= \overline{K}_1 \\ K_2 &= \overline{K}_2 \end{aligned} \qquad \textbf{(12)}$$

We continue to assume that **production functions** are linear homogeneous, twice differentiable, and strictly concave. Thus, perfect competition ensures that equilibrium is characterized by zero economic profits:

$$p_j = a_{Lj} w + a_{Kj} r, \ \forall\, j \in \{1,2\}. \qquad \textbf{(13)}$$



While both production functions are characterized by constant returns to scale, the existence of a fixed factor of production implies that there are **diminishing returns in each sector** (also

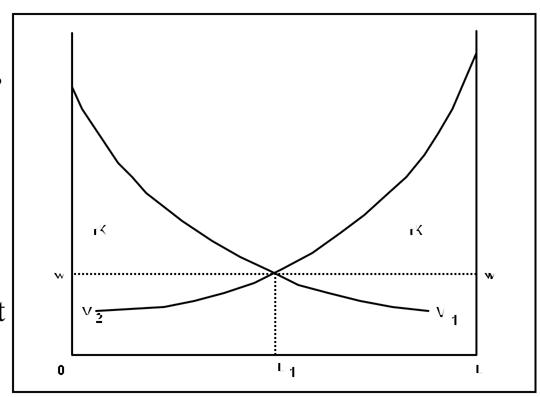
[11]It is not essential to the development of the model that the specific factors be capital, though this will permit us to treat the RV model as a short-run model and the HOS model as a long-run model of the same economy. In fact, it is common to treat the specific-factors as capital and land.

called increasing costs), and therefor *increasing opportunity cost*. As a result, we can use the total product curves along with the labor constraint to derive the production possibility curve. Thus, to increase the output of 1, the economy must reduce the output of 2 production by MPP_{L2}/MPP_{L1} (= a_{L1}/a_{L2}) units. The slope of the production frontier, which gives the opportunity cost of 1 in terms of 2 is thus: $-MPP_{L2}/MPP_{L1}$. It is easy to see that as we move along the L constraint, increasing the output of 1, MPP_{L1} (i.e. the slope of the graph of the production function) falls and MPP_{L2} rises. Thus, we can see that this gives a efficient frontier which is "bowed out" from the origin. This shows that the model we have just constructed is characterized by **increasing opportunity costs** in transforming 2 into 1. In this case it can be seen that a small change in prices need not lead to a discontinuous change in production.

With only one factor mobile, it is particularly easy to illustrate production equilibrium by focussing on the market for labor. Given our assumptions about perfect competition in the L and product markets, we know that factors must be paid the value of their marginal products, i.e.:

$$w = VMP_{L1} := P_1 a_{L1} = P_2 a_{L2} := VMP_{L2}.$$

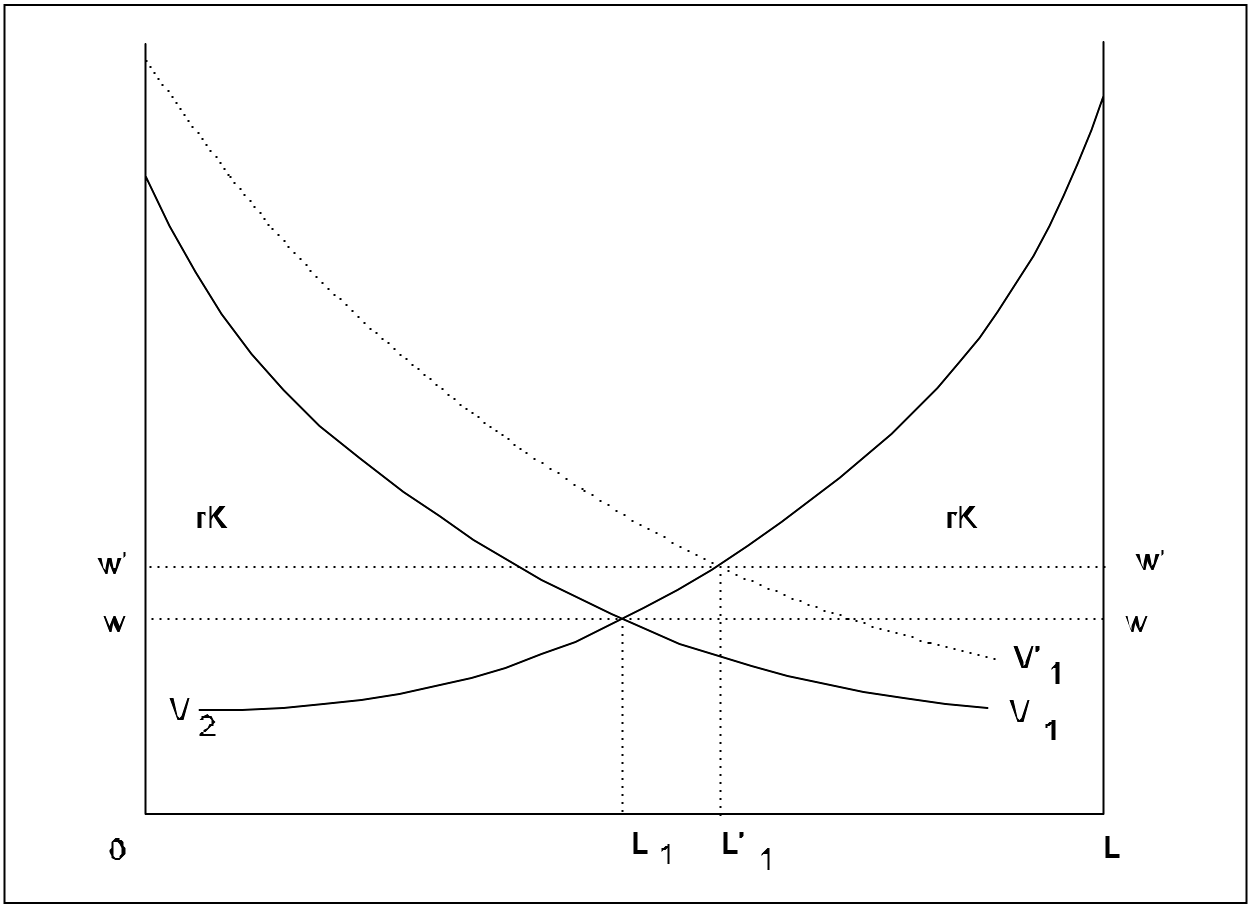
We can plot the VMP_{Lj} noting several relevant facts about the graphs of these functions: they are downward sloping, reflecting our assumption that marginal products are positive but decreasing in L holding K_j constant; their shape and location are determined by the technology, the endowment of K_j, and the output price, p_j. Plotting these curves with respect to the labor endowment $(\bar{L})$ allows us to graphically represent the equilibrium in the factor market. First, note that the equilibrium wage is found at



the intersection of the two $\text{VMP}_{L\,j}$ curves. Second, note that we have identified the allocation of L between the two sectors. This also tells us the output of each of the sectors via the production functions.

Finally, note that we can find the division of national product between the three factors. $w\overline{L}$ gives the payment to L-owning households; and it is a result from calculus that the area under a marginal curve gives the value of total function. Thus the area under the $\text{V}_{L\,j}$ curve gives the value of total output in sector j. Subtracting the payment to L in sector j (i.e. wL_j) leaves the return to K_j. Thus, the "triangular" area above the w-line, below the $\text{V}_{L\,j}$ curve gives the return to K in sector j (i.e. $r\overline{K}_j$, j = 1,2).

Using this graphical apparatus, we now ask how does an increase, say 20%, in P_1 affect these areas. The return to K_2, r_2, has fallen (i.e. the area representing it's income is smaller).



What has happened? P_2 is unchanged, while w has risen, leaving less income to be distributed among the fixed number of units of K_2. Another way of thinking about this is that, with less L_2, the MPP_{K2} has fallen. In any event, r_2 has fallen relative to both commodity prices:

$$\hat{P}_1 > \hat{P}_2 > \hat{r}_2 .$$

This means that the real income (and thus the welfare) of K_2-owning households has fallen. The return to L (i.e. w) has risen, but by less than 20%. A rise of 20% would occur if there was no change in the allocation of L between the 1 and 2 sectors. But, as we have just seen, the increase in the V_{L1}, and the concomitant increase in w, attracts L into the 1 sector. As L enters the 1 sector MPP_{L1} falls, as does the equilibrium wage. Thus, the wage rises relative to P_2, which is unchanged, and falls relative to P_1:[12]

$$\hat{P}_1 > \hat{w} > \hat{P}_2 > \hat{r}_2.$$

Thus, the effect of the shift from autarky to free trade on L-owning households is ambiguous. We would need to know more about the consumption preferences of these HHs to say anything definitive about their welfare. The return to K_1 (i.e. r_1) is greater than 20% We know that if the L allocation between 1 and 2 had stayed the same that both L_1 and K_1 would have risen by 20%. Geometically, this would imply a 20% increase in both of the areas representing factor income. But we have just seen that w rises by less than 20%, so the "triangular" area above the original L allocation must be greater than 20%. In addition, K_1 reaps the additional area above the wage on all but the last unit of new L hired. Thus

$$\hat{r}_1 > \hat{P}_1 > \hat{w} > \hat{P}_2 > \hat{r}_2.$$

K_1-owning HHs gain unambiguously as a result of the shift from autarky to free trade.

In our analysis of the Ricardian economy, our assumption of a single, homogeneous factor of production (L) eliminated any income distribution effects of trade. In the RV economy, as with our exchange economy, we can see how trade differentially affects HH welfare. The standard gains-from-trade result from the basic trade model goes through just a before: the economy as a whole gains in switching from autarky to free trade. Now, however, we see that some HHs can lose unambiguously while the economy as a whole gains. As we will discuss later in the term, this may help us understand

[12]Appendix 2 to this chapter develops the proof of this result more formally.

why some groups resist trade policy changes.

One very convenient interpretation of the relationship between the RV and HOS models is to interpret the first as a short-run version of the second. That is, it is a standard practice to define the long-run economically as the period in which all economically relevant adjustments can be made. Since equilibrium in the HOS model is characterized precisely by all commodity and factor markets being in equilibrium, it is clearly a long-run model. By defining the specific factors in the RV model as sector-specific capitals, we can treat that model as one in which the *K*-market adjustments have not been made. That is, we can treat it as a short-run model of an economy whose long-run equilibrium is characterized by full equilibrium in capital markets (i.e. all units of capital earning the same rental rate)--the HOS model.[13]

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[13]This interpretation is most clearly spelled out in important papers by Mayer (1974) and Neary (1978). The Neary paper also presents an interesting analysis of the transition between the short- and the long-run models.

Appendix I:

Formal Comparative Static Analysis of the HOS Model

The proof of the Stolper-Samuelson theorem follows Jones (1965) classic proof. In this proof we work with the zero-profit conditions for an equilibrium.

$$\begin{aligned} a_{L1} w + a_{K1} r &= p_1 \\ a_{L2} w + a_{K2} r &= p_2 \end{aligned} \qquad \textbf{(1)}$$

Taking the total differential of the zero profit conditions gives:

$$a_{L1}\, dw + w\, da_{L1} + a_{K1}\, dr + r\, da_{K1} = dp_1\ ;$$

$$a_{L2}\, dw + w\, da_{L2} + a_{K2}\, dr + r\, da_{K2} = dp_2\ .$$

Division by p_j gives

$$\frac{a_{L1}}{p_1} dw + \frac{w}{p_1} da_{L1} + \frac{a_{K1}}{p_1} dr + \frac{r}{p_1} da_{K1} = \frac{dp}{p_1} = \hat{p}_1$$

$$\frac{a_{L2}}{p_2} dw + \frac{w}{p_2} da_{L2} + \frac{a_{K2}}{p_2} dr + \frac{r}{p_2} da_{K2} = \frac{dp}{p_2} = \hat{p}_2$$

Manipulate to get proportional changes, i.e. $\hat{x} = \dfrac{dx}{x}$:

$$\frac{a_{L1} w}{p_1} \hat{w} + \frac{a_{L1} w}{p_1} \hat{a}_{L1} + \frac{a_{K1} r}{p_1} \hat{r} + \frac{a_{K1} r}{p_1} \hat{a}_{K1} = \hat{p}_1$$

$$\frac{a_{L2} w}{p_2} \hat{w} + \frac{a_{L2} w}{p_2} \hat{a}_{L2} + \frac{a_{K2} r}{p_2} \hat{r} + \frac{a_{K2} r}{p_2} \hat{a}_{K2} = \hat{p}_2$$

Let 2_{ij} be the share of factor I in the value of output j, i.e. $\theta_{ij} = \dfrac{a_{ij} w_j}{p_j}$:

$$\theta_{L1}\hat{w} + \theta_{L1}\hat{a}_{L1} + \theta_{K1}\hat{r} + \theta_{K1}\hat{a}_{K1} = \hat{p}_1$$
$$\theta_{L2}\hat{w} + \theta_{L2}\hat{a}_{L2} + \theta_{K2}\hat{r} + \theta_{K2}\hat{a}_{K2} = \hat{p}_2$$

Rearranging gives

$$\begin{aligned}\theta_{L1}\hat{w} + \theta_{k1}\hat{r} &= \hat{p}_1 - (\theta_{L1}\hat{a}_{L1} + \theta_{K1}\hat{a}_{K1});\\ \theta_{L2}\hat{w} + \theta_{k2}\hat{r} &= \hat{p}_2 - (\theta_{L2}\hat{a}_{L2} + \theta_{K2}\hat{a}_{K2}).\end{aligned} \quad \textbf{(2)}$$

It is easy to show that, given the competitive and technological assumptions we have made, the expressions in parentheses are equal to zero. First note that the rational entrepreneur seeks to minimize costs (= $a_{Lj}w + a_{Kj}r$). In doing this, the entrepreneur takes factor prices as fixed, and varies the a_{ij}'s so as to set the derivative of cost equal to zero. Totally differentiate the unit isoquant ($1 = f^{\,j}(a_{Lj},a_{Kj})$) to get

$$\frac{\partial f^j(\cdot)}{\partial a_{Lj}}\, d a_{Lj} + \frac{\partial f^j(\cdot)}{\partial a_{Kj}}\, d a_{Kj} = 0.$$

Multiply through by p_j to get

$$w\, da_{Lj} + r\, da_{Kj} = 0.$$

Note that this gives the familiar condition that, in equilibrium, the MRTS = r/w. Written in relative terms, this is

$$\theta_{Lj}\hat{a}_{Lj} + \theta_{Kj}\hat{a}_{Kj} = 0.$$

Thus, we can write (2) in matrix form as

$$\begin{bmatrix} \theta_{L1} & \theta_{K1} \\ \theta_{L2} & \theta_{K2} \end{bmatrix} \begin{bmatrix} \hat{w} \\ \hat{r} \end{bmatrix} = \begin{bmatrix} \hat{p}_1 \\ \hat{p}_2 \end{bmatrix} \qquad \textbf{(3)}$$

This says that in each industry the distributive-share weighted average of factor-price changes equals the relative commodity price change.

We can sign the determinant of the 2-matrix as follows. Note that the distributive shares must sum to one in each sector, which implies that we can rewrite the 2-matrix as

$$\begin{bmatrix} \theta_{L1} & 1 - \theta_{L1} \\ \theta_{L2} & 1 - \theta_{L2} \end{bmatrix}$$

Denoting the determinant of the 2-matrix by *2* (= 2_{L1} - 2_{L2}) we can substitute for the 2_{ij} (= wa_{ij} / p_j) to get

$$\theta_{L1} - \theta_{L2} = \frac{a_{L1} w}{p_1} - \frac{a_{L2} w}{p_2}$$

Now use $p_j = a_{Lj}w + a_{Kj}r$ to get

$$\begin{aligned} |\theta| &= w \left(\frac{a_{L1}(a_{L2} w + a_{K2} r) - a_{L2}(a_{L1} w + a_{K1} r)}{p_1 p_2} \right) \\ &= \frac{w r}{p_1 p_2} \left(a_{L1} a_{K2} - a_{L2} a_{K1} \right) \\ &= \frac{w r a_{K1} a_{K2}}{p_1 p_2} \left(\frac{a_{L1}}{a_{K1}} - \frac{a_{L2}}{a_{K2}} \right). \end{aligned} \qquad \textbf{(4)}$$

Note that this expression is positive if, and only if, sector 1 production is labor-intensive relative to sector 2.

From equation system (3) and Cramer's rule, we can solve for factor-price changes as a function of commodity-price changes. For concreteness, suppose that p_1 rises and p_2 stays constant, then (3) is

$$\begin{bmatrix} \theta_{L1} & \theta_{K1} \\ \theta_{L2} & \theta_{K2} \end{bmatrix} \begin{bmatrix} \hat{w} \\ \hat{r} \end{bmatrix} = \begin{bmatrix} \hat{p}_1 \\ 0 \end{bmatrix} \qquad \textbf{(5)}$$

Cramer's rule gives

$$\hat{w} = \frac{\hat{p}_1 \theta_{K2}}{|\theta|}; \qquad \textbf{(6)}$$
$$\hat{r} = -\frac{\hat{p}_1 \theta_{L2}}{|\theta|}.$$

We want to show that:

$$\hat{w} > \hat{p}_1 > \hat{p}_2 = 0 > \hat{r}. \qquad \textbf{(7)}$$

Suppose that good 1 production is *L*-intensive relative to good 2 production, we know from (4) that *2* = $2_{L1} - 2_{L2} = 2_{K2} - 2_{K1} > 0$. Since $\frac{\theta_{K2}}{\theta_{K2} - \theta_{K1}} > 1$, it is easy to see that $\hat{w} > \hat{p}_1$. However, since $\hat{r} < 0$, and $\hat{p}_2 = 0$, we can easily see that

$$\hat{w} > \hat{p}_1 > \hat{p}_2 = 0 > \hat{r}. \quad \blacksquare$$

This is the **Stolper-Samuelson Theorem**. It is straightforward to extend this result to incorporate $\hat{p}_1 > \hat{p}_2 > 0$.

Appendix II

Formal Comparative Static Analysis of the Specific-Factors Model

In this appendix we follow Jones (1971) to illustrate the comparative static analysis of the specific-factors model, focusing on the effect of a commodity price change on factor returns. As with the HOS model, equilibrium in the the specific-factors model is characterized by full-employment for each of the three factors. If we call the mobile factor labor (*L*) we can write the full-employment conditions as:

$$\begin{aligned} a_{11}y_1 &= \bar{z}_1 \\ a_{22}z_2 &= \bar{z}_2 \\ a_{L1}y_1 + a_{L2}y_2 &= \bar{L}. \end{aligned} \tag{1}$$

We will also need zero-profit conditions. Denoting the payment to the specific-factors by r_j and to the mobile factor by w, we can write these as:

$$\begin{aligned} a_{11}r_1 + a_{L1}w &= p_1 \\ a_{22}r_2 + a_{L2}w &= p_2 \end{aligned} \tag{2}$$

In the HOS model, capital was costlessly and instantaneously mobile between sectors, so $r_1 = r_2 = r$, and the a_{ij}'s were functions of $\{w, r\}$ via the cost minimization that is occurring in the background to this problem. In this case, with commodity prices exogenous by the small open economy assumption, the two zero-profit conditions were sufficient to determine the two factor-prices. Unfortunately, in the specific-factors model z_1 and z_2 are distinct factors, so the zero-profit conditions must be supplemented with information from the full-employment condtions. Thus, we can solve the first two full-employment conditions for the outputs (i.e. $y_j = z_j / a_{jj}$) and substitute in the full-employment condition for the mobile factor to get

$$\frac{a_{L1}}{a_{11}}\bar{z}_1 + \frac{a_{L2}}{a_{22}}\bar{z}_2 = \bar{L}. \tag{3}$$

Equations (2) and (3) are a system of 3 equations to determine the 3 factor prices, $\{r_1, r_2, w\}$, with the two commodity prices and all three endowments as parameters.[14]

Using the method of the previous appendix, we can get the equations of change for the equilibrium relations in (2) as:

$$\begin{aligned} \theta_{11}\hat{r}_1 + \theta_{L1}\hat{w} &= \hat{p}_1 \\ \theta_{22}\hat{r}_2 + \theta_{L2}\hat{w} &= \hat{p}_2 . \end{aligned} \tag{4}$$

Note that, as in the HOS case, the proportional commodity price change is a distributive-share weighted average of the changes in factor-prices. To get equivalent equations of change for equation (3) will take a bit more work. The first step is to recognize that the elasticity of substitution between z_j

[14]We will see in the next chapter that it is a general property of models with more factors than goods that the endowments, as well as commodity prices, are needed to determine the equilibrium factor-prices.

and L_j in sector j is given by:[15]

$$\sigma_j = \frac{\hat{a}_{jj} - \hat{a}_{Lj}}{\hat{w} - \hat{r}}. \tag{5}$$

From this it follows that we can find the change in unit factor inputs as a function of change in factor prices and the substitution elasticity as:

$$\hat{a}_{jj} - \hat{a}_{Lj} = (\hat{w} - \hat{r}_j)\,\sigma_j. \tag{6}$$

Assuming that the mobile factor can change but that there is no change in the endowment of specific-factors of production, differentiate equation (3) to get:

$$d\,a_{L1}\frac{\bar{z}_1}{a_{11}} - a_{L1}\frac{d\,a_{11}}{a_{11}^2}\bar{z}_1 + d\,a_{L2}\frac{\bar{z}_2}{a_{22}} - a_{L2}\frac{d\,a_{22}}{a_{22}^2}\bar{z}_2 = d\,L$$

Now, divide both sides by $\bar{L}$ to get

$$d\,a_{L1}\frac{1}{\bar{L}}\frac{\bar{z}_1}{a_{11}} - \frac{a_{L1}}{\bar{L}}\frac{d\,a_{11}}{a_{11}^2}\bar{z}_1 + d\,a_{L2}\frac{1}{\bar{L}}\frac{\bar{z}_2}{a_{22}} - \frac{a_{L2}}{\bar{L}}\frac{d\,a_{22}}{a_{22}^2}\bar{z}_2 = \frac{d\,L}{\bar{L}} = \hat{L}.$$

Using the definition $\lambda_{ij} = \frac{a_{ij}y_j}{\bar{z}_i}$, and the fact that $a_{jj}y_j = \bar{z}_j$ from equations (1), we can rewrite (8) as:

---

[15]σ_j is the proportional change in the unit input ratio (a_{jj} / a_{Lj}) for a one percent change in the *w*/*r* ratio. Log differentiation yields the result.

$$\hat{a}_{L1}\lambda_{L1} - \lambda_{L1}\hat{a}_{11} + \hat{a}_{L2}\lambda_{L2} - \lambda_{L2}\hat{a}_{22} = \hat{L}$$

$$\lambda_{L1}(\hat{a}_{L1} - \hat{a}_{11}) + \lambda_{L2}(\hat{a}_{L2} - \hat{a}_{22}) = \hat{L}.$$

Now we can use equation (6) to substitute for the terms in parentheses to get

$$\lambda_{L1}\sigma_1(\hat{w} - \hat{r}_1) + \lambda_{L2}\sigma_2(\hat{w} - \hat{r}_2) = \hat{L}.$$

This can also be written as

$$-\lambda_{L1}\sigma_1\hat{r}_1 - \lambda_{L2}\sigma_2\hat{r}_2 + \sum_{j \in J}\lambda_{Lj}\sigma_j\hat{w} = \hat{L}. \qquad \textbf{(7)}$$

We can represent the system given by equations (4) and (7) in matrix form as:

$$\begin{bmatrix} \theta_{11} & 0 & \theta_{L1} \\ 0 & \theta_{22} & \theta_{L2} \\ -\lambda_{L1}\sigma_1 & -\lambda_{L2}\sigma_2 & \sum_{j \in J}\lambda_{Lj}\sigma_j \end{bmatrix} \begin{bmatrix} \hat{r}_1 \\ \hat{r}_2 \\ \hat{w} \end{bmatrix} = \begin{bmatrix} \hat{p}_1 \\ \hat{p}_2 \\ \hat{L} \end{bmatrix}. \qquad \textbf{(8)}$$

In more compact form this is:

$$\Theta\hat{r} = \hat{p}. \qquad \textbf{(9)}$$

If we hold the endowment of labor fixed, to focus on the effect of commodity price changes on returns to factors, the last element in the vector on the left hand side will be a zero. To solve this system using Cramer's rule we will need an expression for the determinant of 1:

$$
\begin{aligned}
\det(\Theta) &= \theta_{11}\,\theta_{22}\,(\lambda_{L1}\sigma_1 + \lambda_{L2}\sigma_2) + \theta_{L1}\,\theta_{22}\,\lambda_{L1}\sigma_1 + \theta_{11}\,\theta_{L2}\,\lambda_{L2}\sigma_2 \\
&= [\,\theta_{11} + \theta_{L1}\,](\theta_{22}\,\lambda_{L1}\sigma_1) + [\,\theta_{22} + \theta_{L2}\,](\theta_{11}\,\lambda_{L2}\sigma_2) = \theta_{22}\,\lambda_{L1}\sigma_1 + \theta_{11}\,\lambda_{L2}\sigma_2 \\
&= \theta_{11}\,\theta_{22}\left[\lambda_{L1}\frac{\sigma_1}{\theta_{11}} + \lambda_{L2}\frac{\sigma_2}{\theta_{22}}\right].
\end{aligned}
\tag{10}
$$

Note that, for the third equality, we use the fact that $2_{jj} + 2_{Lj} = 1$ (i.e. the distributive shares for a given sector must sum to 1). Suppose that we denote the matrix created by substituting $\hat{p}$ for the k'th column of 1 by 1^k. Then to calucuate the value of the k'th element of $\hat{r}$ using Cramer's rule:

$$
\hat{r}_k = \frac{\det(\Theta^k)}{\det(\Theta)}.
$$

Thus, to find the effect of changes in commodity prices on the return to labor, we need an expression for det(1^3):

$$
\begin{aligned}
\det(\Theta^w) &= \hat{p}_1\,\theta_{22}\,\lambda_{L1}\,\sigma_1 + \hat{p}_2\,\theta_{11}\,\lambda_{L2}\,\sigma_2 \\
&= \theta_{11}\,\theta_{22}\left[\hat{p}_1\,\lambda_{L1}\,\frac{\sigma_1}{\theta_{11}} + \hat{p}_2\,\lambda_{L2}\,\frac{\sigma_2}{\theta_{22}}\right].
\end{aligned}
\tag{11}
$$

In the expressions for both determinants in (10) and (11) we find terms like $\frac{\sigma_j}{\theta_{jj}}$. Mussa (1977) discusses this term in detail: it is the elasticity of the marginal product curve of the mobile factor in sector j. Using Cramer's rule, we divide the result in (11) by that in (10) to get

$$\hat{w} = \sum_{j \in J} \beta_j \hat{p}_j \quad \text{where} \quad \beta_j = \frac{\lambda_{Lj} \dfrac{\sigma_j}{\theta_{jj}}}{\sum_{k \in J} \lambda_{Lk} \dfrac{\sigma_k}{\theta_{kk}}} \tag{12}$$

It should be clear that $\$_j > 1$ and $\sum_{j \in J} \beta_j = 1$.

Similarly, to find an expression for the return to, say, the specific-factor in sector 1 we need to find:[16]

$$\begin{aligned} \det(\Theta^1) &= \hat{p}_1 \theta_{22} (\lambda_{L1} \sigma_1 + \lambda_{L2} \sigma_2) + \hat{p}_1 \theta_{L2} \lambda_{L2} \sigma_2 - \hat{p}_2 \theta_{L1} \lambda_{L2} \sigma_2 \\ &= [\theta_{22} + \theta_{L2}](\hat{p}_1 \lambda_{L2} \sigma_2) + \hat{p}_1 \theta_{22} \lambda_{L1} \sigma_1 - \hat{p}_2 \theta_{L1} \lambda_{L2} \sigma_2 \\ &= \hat{p}_1 \left[\theta_{22} \lambda_{L1} \sigma_1 + \lambda_{L2} \sigma_2 \right] - \hat{p}_2 \theta_{L1} \lambda_{L2} \sigma_2 \\ &= \theta_{11} \theta_{22} \left\{ \hat{p}_1 \left[\lambda_{L1} \frac{\sigma_1}{\theta_{11}} + \frac{1}{\theta_{11}} \frac{\sigma_2}{\theta_{22}} \lambda_{L2} \right] - \frac{\theta_{L1}}{\theta_{11}} \lambda_{L2} \frac{\sigma_2}{\theta_{22}} \hat{p}_2 \right\}. \end{aligned} \tag{13}$$

Division of equation (13) by equation (10) gives the effect of price changes on the return to specific-factor 1.

Now consider the exercise from the text of raising only a single commodity price, say p_1, by one percent. It should be clear from equation (12) that the return to labor will rise. Making the appropriate adjustments to equation (13) we can see that the return to the specific factor in sector 2 will fall. Finally, from equation (13) we can see that the return to the specific factor in sector 1 must rise. Furthermore, from equations (11) and (13), recognizing that $\hat{p}_2 = 0$, we get

[16]The return to the specific factor in sector 2 is found in exactly the same way.

$$\lambda_{L1}\frac{\sigma_1}{\theta_{11}} + \frac{1}{\theta_{11}}\frac{\sigma_2}{\theta_{22}}\lambda_{L2} > \lambda_{L1}\frac{\sigma_1}{\theta_{11}} \rightarrow \hat{r}_1 > \hat{w}. \qquad \textbf{(14)}$$

From the weighted average property of price changes given in equations (4), which implies that commodity price changes must be trapped between factor-price changes, we have our result that:

$$\hat{r}_1 > \hat{p}_1 > \hat{w} > \hat{p}_2\ (= 0) > \hat{r}_2.$$

As with the HOS case, this magnification effect carries over to the case in which both commodity prices change by different proportional amounts.