Chapter V:

Generalizing the Neoclassical Production Model

> It is one thing to say that, given any initial equilibrium position, there exists a one-to-one association between commodities and factors such that a change in any commodity price will lead to a more than proportionate change (in the same direction) in the corresponding factor price. It is quite another thing to state that it is possible to find a one-to-one association between goods and factors *in advance* such that, starting from any equilibrium, a change in any commodity price will lead to a more than proportionate change in the price of the *already specified* factor. (Chipman, 1969, pg. 399, emphasis in original)

In the previous chapter we introduced the basic model that will serve to characterize the economic component of our political economy model: the 2×2 HOS model. Due to the low dimensionality of this model, we are able to clearly see the effects of various types of policy changes. In particular, the Stolper-Samuelson theorem provides a strong link between policies that change commodity prices (like tariffs and other trade barriers) and household welfare (through the effect of factor-prices on household income). However, it should be quite clear that this tractability comes at some considerable cost--the assumed low dimensionality is clearly falsified by the most casual empiricism. Thus, it is important to ask how sensitive the results of the HOS model (especially the Stolper-Samuelson theorem which is so central to our analysis) are to alternative assumptions about dimensionality. Because the Stolper-Samuelson theorem is so intuitively appealing, it is important to understand the way that the theorem depends on the assumptions of the model. As Jones and Scheinkman (1977) emphasize, the HOS model is special in two ways: the dimensionality is low; and the number of factors is equal to the number of goods. In this chapter we review some of the results on generalizing the Stolper-Samuelson theorem to higher dimensional cases.[1] The next section presents the general $m \times n$ neoclassical model of production.[2] This is followed by a development of a fairly general comparative static framework for this model. The majority of the chapter then develops the Stolper-Samuelson theorem in the context of this framework.

The *m* × *n*, Neoclassical Production Model

In generalizing the neoclassical model to the case of arbitrary numbers of factors and commodities we will retain as much of the structure used in the 2×2 case as possible. Thus, we continue to abstract from such potentially important aspects of reality as intermediate goods, joint production, increasing returns to scale, and market distortions.[3] In addition to retaining all of our institutional assumptions from the preceding chapter, we will continue to assume that technology can be characterized by production functions that are positive, linearly homogeneous, twice smoothly

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[1] Because we will be concerned primarily with development of a key comparative static property of the model, appendix II provides a discussion of the method of comparative statics and the role of the implicit function theorem in such analysis. Appendix I provides a review of some basic terminology and results from linear algebra.

[2] To be clear on notation: I is the set of all factors, the general index for a factor is i, and there are a total of m factors. Similarly, J is the set of all commodities; j is a general index for commodities; and there are n total commodities. Thus, a_{ij} is the input of factor i required to produce one unit of commodity j.

[3] Trade theoretic research has developed generalization of the standard model to all of these cases. Because we will not be using such generalizations in our political-economy modeling, we do not develop these analyses in detail. In the last section of this chapter, however, we will briefly discuss the results of this research.

differentiable, and strictly concave in inputs, where inputs are now given by an m-vector $\boldsymbol{z}$: $y_j = f^j(\boldsymbol{z}_j)$.[4] We will continue to assume that no output can be produced without strictly positive input. However, now that we have an arbitrary number of factors of production, we need to be more specific about patterns of factor use. In the HOS model we assumed that both factors were used in strictly positive quantities by both productive sectors. One way of generalizing the model would be to assume that all sectors use strictly positive amounts of all factors, but, while we will adopt it as a special case later in this chapter, this assumption seems a bit extreme. Instead, we will adopt the weaker assumption that every sector uses at least two factors and every factor is used in at least two sectors.[5]

Because of constant returns to scale, we can focus on cost minimization relative to the unit isoquant. Furthermore, it is a standard result of modern production theory that, given the assumptions we have made on technology, we can represent the technology of sector j with a *unit cost function*:

$$c^j(\mathbf{w}) := \min_{\mathbf{z}} \left\{ \mathbf{w} \bullet \mathbf{z} \,\middle|\, f^j(\mathbf{z}) \geq 1, \mathbf{z} \geq 0 \right\}.$$

Under the assumption that production functions are positive, linearly homogeneous, and strictly concave for $\boldsymbol{z} > 0$, the unit cost function is also positive, linearly homogeneous, and strictly concave for $\boldsymbol{w} \geq 0$.[6]

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[4]Recall that our institutional assumptions are that: all agents (i.e. households and firms) are rational; all markets are complete and perfect; and the national economy is small in world markets.

[5]Note first that, in the 2 × 2 case this yields the usual HOS structure. It should be noted, however, that many standard models make different assumptions. In particular, the Ricardo-Viner model assumes that all sectors use two-factors, but that one is common to all sectors (usually labor), and the other is specific to a single sector.

[6]See Diewert (1982) or Woodland (1982, chapter 2) for properties of cost functions. For vector inequalities, we will use $\boldsymbol{x} \gg 0$ to mean that every element x_i of the vector is greater than zero; $\boldsymbol{x} > 0$ to mean that at least some x_i is greater than zero; and $\boldsymbol{x} \geq 0$ to mean that every x_i is greater than or

We show in appendix III that the Hessian matrix of $c^j(\boldsymbol{w})$, c^j_{ww}, is symmetric, negative semidefinite, and has rank equal to m - 1, where m is the number of factors used in the production of good j.

For our present purposes, the most important property of the unit cost function is given by *Shepard's lemma* which asserts that the amount of factor i needed for one unit of output of good j can be found as the derivative of the unit cost function with respect to the cost of factor i. That is:

$$\frac{\partial c^j(\mathbf{w})}{\partial w_i} = a_{ij}(\mathbf{w}).$$

Thus, if the unit cost function for good j is twice differentiable, then the $a_{ij}(\boldsymbol{w})$ are unique, differentiable functions of $\boldsymbol{w}$. Furthermore, since the $c^j(\boldsymbol{w})$ are homogeneous of degree 1, the a_{ij} are homogeneous of degree zero in $\boldsymbol{w}$.

As a result of Shepard's lemma, we can find the vector of unit factor demands as the gradient of the unit cost function: $\nabla c^j(\boldsymbol{w}) = \boldsymbol{a}^j(\boldsymbol{w}) \equiv (a_{1j}, a_{2j}, \ldots a_{lj})'$. Thus, the technology for the economy as a whole can be given by the $m \times n$ matrix

$$A(\mathbf{w}) \coloneqq \left[a_{ij}(\mathbf{w})\right],$$

each of whose columns gives the technology for one of the n industries. Assuming sufficient substitutability in production and denoting the n-vector of sectoral outputs as $\boldsymbol{y} \equiv \{y_1, \ldots, y_n\}$, we can write the *full-employment equilibrium conditions* as:

$$A(\mathbf{w})\mathbf{y} = \overline{\mathbf{z}}. \qquad \textbf{(1)}$$

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equal to zero. Thus, $\boldsymbol{z}^j > 0$ reflects our assumption that positive production of good j requires positive input of at least some (in fact, at least 2) factor(s) of production.

As in the 2×2 case, the endowment vector $\left(\bar{\mathbf{z}}\right)$ and the commodity-price vector (***p***) are exogenous to the small, open economy, while the output vector (***y***) and the factor-price vector (***w***) are determined endogenously. From Euler's theorem applied to a linear homogeneous function, we can write the *zero profit equilibrium conditions* as:[7]

$$A'\left(\mathbf{w}\right)\mathbf{w} \geq \mathbf{p}. \qquad \textbf{(2)}$$

where the prime denotes that the matrix has been transposed.

Focusing on the Stolper-Samuelson and Rybczynski theorems, we start by recalling from the previous chapter that in the 2 × 2 case these results take a very strong form: we can identify a friend and an enemy for every good in physical quantities (Rybczynski) and every factor in prices (Stolper-Samuelson); that identification is based entirely on relative factor-intensity; and there is magnification. With more than two factors of production, unless we are willing to impose a very peculiar production structure, it should be clear that factor-intensity is no longer one-dimensional. So the simple link between factor-intensity and comparative static effects will be lost. Dimensionality greater than 2 × 2 also creates problems for both "friends and enemies" and "magnification", except in the case in which $m = n$.

To understand the problems raised by dimensional generalization, consider the role of the *A*-matrix in equation (1). Suppose that the economy with which we are concerned is a small, open economy, so the price vector (***p***) is fixed and, for the reasons we discussed in the previous chapter, so

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[7] Euler's theorem for homogeneous functions, applied to the (homogeneous of degree one) unit cost function, states that $\nabla c^j(\boldsymbol{w}) \cdot \boldsymbol{w} = c^j(\boldsymbol{w})$, and zero-profits means that $p_j = c^j(\boldsymbol{w})$.

is the optimal technology choice for each industry. In equation (1), the A-matrix takes n-dimensional vectors of outputs ($\boldsymbol{y}$) into an m-dimensional input space (m-space). If we assume that the economy possesses strictly positive quantities of every factor, this economy's endowment $(\bar{\mathbf{z}})$ is a strictly positive vector in m-space. Each of the column's in A defines a vector in m-space and, if all goods are to be produced, the endowment vector must lie in the subspace of m spanned by the technology vectors.[8] With fixed technologies, if this is to be anything but a fluke, the dimensionality of the subspace spanned by the technologies must be equal to the number of factors. That is, there must be at least as many distinct technologies as factors.[9] If this condition holds, then the endowment vector can be changed arbitrarily within a small neighborhood of $\bar{\mathbf{z}}$ without requiring changes in technology and, thus, factor-prices. The importance of this, as we shall see below, is that we will not need the endowment vector to find the effect of a commodity-price change on factor-prices.

Now suppose that $n > m$. We know that the rank of A cannot be greater than m, so at least $n - m$ sectors must have technologies that are not distinct. If we assume that there are m distinct technologies, so that the R(A) = m, simple equation counting tells us that there are an infinite number of output vectors $\boldsymbol{y}$ that satisfy the full-employment conditions in (1). The non-uniqueness is reflected in

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[8]The technology vectors constitute what is called a *convex cone* in m-space. A subset C of ú^{m} is a convex cone if it is closed under addition and under multiplication by positive scalars. That is, if vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are in C, then $\boldsymbol{x} + \boldsymbol{y}$ is also in C; and if $\boldsymbol{x}$ is in C and 8 $ 0, then 8$\boldsymbol{x}$ is in C. For the reasons developed in the above text, this particular cone is often referred to as the *cone of diversification*.

[9]By "distinct" we mean that the column vector representing a given sector's technology cannot be represented as a linear combination of any other technologies.

the fact that the feasible set of outputs contains “flats”.[10] Because the solution of (1) for outputs does not yield a unique solution, it should be clear that the tight link between a change in endowments and a change in outputs established in the Rybczynski theorem for the 2×2 case will not exist in the $n > m$ case.

The real problem with the $n > m$ case, however, has more to do with the zero-profit conditions (2) than the full-employment conditions. Recall that the zero-profit conditions involve A' taking m-dimensional vectors of factor-prices into n-dimensional commodity-price space. In the case of equations (1), when $m > n$, the simple existence of m factors of production lead us to expect a change in factor-prices. All this means is that factor-prices are then functions of both endowments and commodity-prices.[11] The $n > m$ case is, by contrast, is characterized by a strong presumption of instability for the small open economy. From an initial equilibrium in which $n > m$ commodities are produced, any change in the price vector is likely to cause some sectors to cease production. That is, in an open economy, even if consumers in the economy demand positive quantities of all n commodities, there is no reason why they should all be produced in the country. As a result, much of the work on comparative statics focuses on the $m \geq n$ case. In any event, it should be clear that moving to higher dimensions for both goods and factors of production is going to upset the strong, simple structure of the Stolper-Samuelson and Rybczynski results.

Broadly speaking, there have been three major approaches to developing high dimensional

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[10] We can see this easily in the two-good, one-factor Ricardian model.

[11] Consider the case of one good being produced with two factors. The endowment vector picks out the relevant input ratio, which determines marginal physical products, which along with the commodity price determines the factor-price.

generalizations of the main comparative statics for the Heckscher-Ohlin-Samuelson model.[12] One approach attempts to find sufficient conditions on the A-matrix such that results like the Stolper-Samuelson and Rybczynski theorems hold. Such generalizations are possible only when $m = n = R(\mathrm{A})$, though this is far from sufficient. While this literature has provided great insight into the mathematical structure of trade models, it seems fair to say that little in the way of intuitively appealing economic insight has been generated, while the assumptions on the structure necessary to develop any results at all are extremely strong. An alternative approach jettisons both magnification and explicit identification of gainers and losers to get results that identify friends and enemies *on average*. This has proven a very successful strategy, but for political-economy modeling it is precisely the magnification and, to a lesser degree, the explicit identification of friends and enemies that makes the Stolper-Samuelson theorem so useful. As a result, we now turn to a more detailed development of an approach that focuses explicitly on magnification in a relatively general analytical environment. We will follow, in particular the development of this approach due to Ronald Jones and José Scheinkman (1977).

Comparative Statics Framework for the $m \times n$ Production Model[13]

In this section we develop the framework within which the comparative static propositions of

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[12]See Ethier (1981) for a truly magisterial survey of the full range of issues associated with dimensional generalization. Deardorff and Stern (1994) contains not only the original statement of the Stolper-Samuelson theorem, but a number of the major generalizations.

[13]The discussion in this section is drawn primarily from Jones and Scheinkman (1977). Also fundamental are the papers by Diewert and Woodland (1977) and Chang (1979). Jones and Easton (1983) develop the economic intuition behind such models using a general 3-factor × 2-commodity model, beautifully illustrating the complex interactions between factor intensity and factor substitution that make determinate comparative static results difficult to come by.

the $m \times n$ neoclassical production model can be derived. Assuming that the equilibrium exists, we need to totally differentiate the system characterizing that equilibrium given in equations (1) and (2).[14] We will consider each in turn and then collect the results together in the Jacobian matrix for the system as a whole that contains the comparative static results. The following section will examine the Stolper-Samuelson theorem in some detail.

We begin by totally differentiating the full-employment conditions given in (1). Since a representative equation of this system is $\sum_{j\in J} a_{ij}y_j = \bar{z}_i$, for a small change in the i'th factor this procedure yields

$$\sum_{j\in J} y_j da_{ij} + \sum_{j\in J} a_{ij} dy_j = dz_j. \tag{3}$$

However, since the $a_{ij}(\boldsymbol{w})$ are differentiable, and letting $a_{ij}^k := \dfrac{\partial a_{ij}}{\partial w_k}$, we can get

$$da_{ij} = \sum_{k\in I} \frac{\partial a_{ij}}{\partial w_k} dw_k \equiv \sum_{k\in I} a_{ij}^k dw_k. \tag{4}$$

Now, use equation (4) to substitute for da_{ij} in (3) to get

$$\sum_{k\in I} s_{ik} dw_k + \sum_{j\in J} a_{ij} dy_j = dz_i. \tag{5}$$

In writing (5), we use the definition $s_{ik} = \sum_{j\in J} y_j a_{ij}^k$, which shows how economy-wide demand for

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[14]Using the classic result of Debreu (1959, pp. 83-84) it is straightforward to show that an equilibrium exists for an economy with the properties we have assumed (conditional on preferences which, like our assumptions on production, that satisfy Debreu's conditions). Given existence of an equilibrium, differentiability of the $a_{ij}(\boldsymbol{w})$, and a nonzero determinant for the Jacobian matrix of the entire system, the conditions for application of the implicit function theorem are satisfied, permitting the use of standard comparative static methodology.

factor i would respond at unchanged outputs as the wage of factor k rises. That is, s_{ik} refers to the aggregate effect of movements around *given* isoquants in every sector induced by a change in w_k. Factors i and k are said to be *aggregate substitutes* if $s_{ik} < 0$ and *aggregate complements* if $s_{ik} > 0$ (see appendix III). Thus, the differentiated full-employment conditions in matrix form are:

$$S d\mathbf{w} + A d\mathbf{y} = d\bar{\mathbf{z}}. \qquad \textbf{(6)}$$

In equation (6) $S = [s_{ik}]$ is an $m \times m$ matrix, $d\boldsymbol{w}$ is the m-element column vector of factor wages, and $d\boldsymbol{y}$ is the n-element column vector of outputs. Appendix III refers to S as the *economywide substitution matrix* and following Chang (1979) and Takayama (1982), argues that S is: symmetric, negative definite, with $S\boldsymbol{w} = \boldsymbol{w}'S = 0$, and $R(S) = m - 1$.

Now consider the zero-profit conditions. Assuming that all industries operate at positive levels, we have equations (2). With technology sufficiently flexible to ensure full-employment, we now assume that all factor prices are strictly positive. As in the 2 × 2 case, cost minimization in each industry ensures that

$$\sum_{i \in I} w_i da_{ij} = 0.$$

Thus, in matrix form, differentiation of the zero-profit conditions gives:

$$A' d\mathbf{w} = d\mathbf{p}. \qquad \textbf{(7)}$$

We can collect equations (6) and (7) together in the system

$$\begin{bmatrix} S & A \\ A' & 0 \end{bmatrix} \begin{bmatrix} d\mathbf{w} \\ d\mathbf{y} \end{bmatrix} = \begin{bmatrix} d\mathbf{z} \\ d\mathbf{p} \end{bmatrix}. \tag{8}$$

and, for easy reference, we will let the partitioned matrix on the right-hand side be

$$B = \begin{bmatrix} S & A \\ A' & 0 \end{bmatrix}. \tag{9}$$

B is the Jacobian matrix of the system in (1) and (2). Note that, since S is $m \times m$ and A is $m \times n$, B is an $(m + n) \times (m + n)$ square matrix. Thus, B will be invertible if it is of full rank (i.e. if $R(B) = m + n$). It should also be clear that, since S is symmetric, B is symmetric. This will be a useful fact since it is known that the inverse of a symmetric matrix is itself symmetric.

As suggested in appendix II, the standard approach to comparative static analysis makes extensive use of the implicit function theorem, which in turn relies on invertibility of B. Thus, the next order of business is to determine under what conditions we can be sure that this is the case. Here we can avail ourselves of another result from Chang (1979, lemma 1): the matrix B is nonsingular if and only if $R(A) = n$. Since A is $m \times n$, Chang's lemma suggests that B will be non-singular, and thus invertible, if $m \geq n$. That is, the number of factors must be at least as great as the number of sectors.[15] This means that each sector will be technologically distinct--no sector's input-coefficient vector can be represented as a linear combination of a subset of the remaining sectors' coefficient vectors.

Now suppose that $m \geq n$ so that B is, in fact, invertible. To solve for comparative static results,

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[15]Chang (1979, section 5) presents an analysis of the case in which $n > m$, but this requires greater complexity, without advancing the main concerns of this chapter.

we will need the B^{-1} matrix. We write the inverse of the B matrix as follows[16]

$$B^{-1} = \begin{bmatrix} S & A \\ A' & 0 \end{bmatrix}^{-1} := \begin{bmatrix} K & G \\ G' & L \end{bmatrix} \quad \textbf{(10)}$$

Using (10) and solving (8) for the ($d\mathbf{w}$, $d\mathbf{y}$) vector, we obtain

$$\begin{bmatrix} d\mathbf{w} \\ d\mathbf{y} \end{bmatrix} = \begin{bmatrix} K & G \\ G' & L \end{bmatrix} \begin{bmatrix} d\mathbf{z} \\ d\mathbf{p} \end{bmatrix}. \quad \textbf{(11)}$$

It should be clear that submatrix K shows the impact of a change in the endowment vector on factor-prices when commodity prices are held constant; the submatrix G shows how a change in commodity prices affects factor-prices with given factor-endowments; the submatrix G' links endowment changes to output changes at constant commodity prices; and the submatrix L links output changes along a given transformation surface to commodity price changes. Thus, we can also write B^{-1} as:

$$B^{-1} = \begin{bmatrix} S & A \\ A' & 0 \end{bmatrix}^{-1} := \begin{bmatrix} K & G \\ G' & L \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \mathbf{w}}{\partial \mathbf{z}} & \dfrac{\partial \mathbf{w}}{\partial \mathbf{p}} \\ \dfrac{\partial \mathbf{y}}{\partial \mathbf{z}} & \dfrac{\partial \mathbf{y}}{\partial \mathbf{p}} \end{bmatrix}. \quad \textbf{(12)}$$

We can consider each of the elements of B^{-1} in turn.[17]

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[16]Diewert and Woodland (1977) derive B^{-1} directly from the maximum value function for national product in a production model that permits both interindustry flows and joint production. Chang (1979) and Takayama (1982) use this approach for the model we use here, explicitly linking Diewert and Woodland's framework to that used by Jones and Scheinkman (1977).

[17]Note that symmetry of B^{-1} gives us the *Samuelson reciprocity relations* (Samuelson, 1953-1954) between the Stolper-Samuelson (G) and Rybczynski (G') matrices that have been so effectively exploited in the analysis of high-dimensional production models.

It can be shown that the submatrix K is symmetric and negative semidefinite, with rank $m - n$.[18] Since the rank of the K-matrix is $m - n$, it should be clear that in the even (i.e. $m = n$) case the K-matrix reduces to a zero matrix. This is the *factor-price equalization* result: if two countries with the same technology sets are in the same n-dimensional cone of diversification, and face identical commodity prices, they will have identical factor-prices if, and only if, $m = n$.[19] But when $m > n$ factor-endowments will generally have an effect on factor-prices (as in the two-factor, one-good case mentioned in footnote 11). Furthermore, Jones and Scheinkman (pg. 927) show that, unless a dimensionality condition on the rows of the A matrix is met, then, with $m > n$, an increase in a factor-endowment need not lower (although it cannot raise) that factor's price at constant commodity-prices.

Now consider the submatrix L that gives output changes in response to shifts in commodity prices. Chang (Theorem 2) shows that this submatrix is positive semidefinite with a rank of $n - 1$. Furthermore, Jones and Scheinkman (page 298) use the fact that the rank of S is $m - 1$, to show that L has a *positive diagonal* whenever $n \geq 2$. Thus, as we would expect, a rise in any commodity price must increase the output of that commodity.

The G-matrix reveals how a change in commodity prices affects factor-prices as of given

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[18]This result and its proof can be found in Diewert and Woodland (1977; appendix lemma 3) and in Chang (1979, theorem 2).

[19]An interpretation which is empirically more compelling refers to a single country rather than two trading countries. Leamer (1995) calls to this the *factor-price insensitivity theorem*: a change in a country's endowment vector, **z**, with unchanged technology and unchanged commodity prices, that stays within the cone of diversification, will leave factor-prices unchanged. As with factor-price equalization, this result depends on $m = n$–i.e. on K being a zero matrix. Gaston and Nelson (2001) use this result to help make sense of the empirical result that immigration seems to have had little effect on relative wages, even in local labor markets experiencing large immigration shocks. This suggests a puzzle for political economy which Gaston and Nelson (2000) pursue in more detail.

factor-endowments (i.e. the Stolper-Samuelson relations). The *GN*-matrix links endowment changes (at constant commodity prices) to output changes (the Rybczynski relations). Since we will find it more convenient in our later analysis to work with elasticities, we now carry out our discussion of the Stolper-Samuelson theorem in terms of proportional changes.

Real Income Effects of Price Changes: Generalizations of Stolper-Samuelson

In this section we use the framework developed in the previous section to study the effect of a price change on the real return to factors of production. As we shall see in the next chapter, this information plays a fundamental role in political-economy modeling. As in the 2 × 2 case, it will prove move convenient to work in proportional changes. Thus, we follow the procedure used in the appendix to chapter 2-1 to rewrite the basic framework (8) in terms of relative changes. In matrix form, this is

$$\begin{bmatrix} \Sigma & \Lambda \\ \Theta' & 0 \end{bmatrix} \begin{bmatrix} \hat{\mathbf{w}} \\ \hat{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{z}} \\ \hat{\mathbf{p}} \end{bmatrix}. \qquad \textbf{(13)}$$

E is the $m \times m$ matrix whose characteristic element is $s_{ik} = \frac{w_k s_{ik}}{z_i}$. That is F_{ik} is an economy-wide substitution elasticity showing the percentage change in the economy's use of factor *i* (at unchanged outputs) associated with a 1% increase in the *k*th factor price. The E-matrix has a negative diagonal and *zero row sums.* 7 is the $m \times n$ matrix whose characteristic element is $l_{ij} = \frac{a_{ij} y_j}{\overline{z}_i}$, the share of the economy's endowment of factor *i* in use in sector *j*. Since all 8_{ij} $ 0 and $\sum_{j \in J} l_{ij} = 1,$ 8 is a nonnegative, row-stochastic matrix.[20] 1 is the $m \times n$ matrix whose characteristic element is

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[20]A *row stochastic matrix* is a non-negative matrix in which each row sums to unity. It is also the case that the inverse of a row stochastic matrix, if it exists, also has unit row sums. Chipman (1969,

$\theta_{ij} = \frac{w_i a_{ij}}{p_j}$, the distributive share of factor i in the income from selling commodity j. Because all $\theta_{ij} \geq 0$ and $\sum_{i \in I} \theta_{ij} = 1$, Θ is non-negative and column stochastic. Since Θ' is row-stochastic, the entire matrix is row-stochastic. Furthermore, the full matrix is square and is invertible (i.e. non-singular) if B is invertible (i.e. if $m \geq n$). For ease of reference, we will denote this matrix #.

Our comparative static analysis focuses on the properties of the inverse matrix

$$\begin{bmatrix} \hat{\mathbf{w}} \\ \hat{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} M & N' \\ Q & R \end{bmatrix} \begin{bmatrix} \hat{\mathbf{z}} \\ \hat{\mathbf{p}} \end{bmatrix}. \qquad \textbf{(14)}$$

For generalizations of the Stolper-Samuelson theorem, we will be working with the $n \times m$ sub-matrix N, whose characteristic element n_{ji} gives the effect of a change in p_j on w_i. For the Stolper-Samuelson theorem to hold, there should be some numbering of goods and factors such that all diagonal elements (n_{ji} such that $i = j$) are greater than unity. That is, a one percent increase in the price of good j leads to a greater than one percent increase in the return to factor i. Chipman (1969) calls this the *weak Stolper-Samuelson criterion*. We will say that factor i and good j are *unambiguous friends*.[21] If every row of N contains a negative element n_{jk} we will say that factor k and good j are unambiguous enemies.[22] If all diagonal elements are positive and all off-diagonal elements are negative, then the N

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pg. 402) proves both that Θ is (row) stochastic and that the inverse of a row stochastic matrix has unit row sums.

[21] The use of "unambiguous" as a modifier referring to the magnification effect comes from Ethier (1974).

[22] Note that n_{kj} need not be less than -1, it only needs to be negative. Why?

matrix meets Chipman's *strong Stolper-Samuelson criterion*.[23] That is, in this case, a price change divides all factors into unambiguous winners and losers.

Before analyzing whether we can assert that there are unambiguous friends and enemies in the high dimensional case, we can now make an important point suggested in the quotation from Chipman that begins this chapter. Under the usual assumptions of the 2×2 HOS model, not only are there unambiguous friends and enemies, but those associations are *global* properties of the economy. That is, under the assumption of no factor intensity reversals, one good will be K-intensive at all prices that involve production of both goods and the other good will always be labor intensive. If this condition holds between factor i and good j in the high dimensional case, we will say that they are *globally unambiguous friends* (or *enemies*). Unfortunately, the existence of such global results requires the existence of extremely strong structure on $A(\boldsymbol{w})$.[24] More generally, as many have pointed out, even if all factors and commodities can be linked as unambiguous friends (i.e. if $n_{ji} > 1$ for all $i = j$), the particular factors and commodities linked in that fashion is a property of the equilibrium--that is, these relations are local.

Now we are ready for the main event: under what conditions will there be (locally) unambiguous friends and enemies? Furthermore, rather than pursue the strong Stolper-Samuelson case where price changes divide factors into winners and losers, we will ask a weaker question: does every

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[23]A matrix with this sign pattern is called a *Minkowski matrix*. A useful collection of definitions and results on matrices of the sort used in this kind of analysis can be found in Kemp and Kimura (1978, chapter 3).

[24]See Uekawa, Kemp and Wegge (1973). To get some idea of how strong these conditions are, we simply note that: $A(\boldsymbol{w})$ must be square (i.e. $m = n = R(A)$); and all factors must be used in the production of all commodities. These are only necessary.

factor have an unambiguous friend and an unambiguous enemy in the Stolper-Samuelson sense? The analytical framework for answering this question was first developed by Ethier (1974) for the case in which $m = n = R(A)$ and all commodities must use all factors of production. In this case, Θ is a strictly positive, column stochastic, square matrix that does possess an inverse, N. Thus, denoting the identity matrix I, we can write

$$N\Theta = \mathrm{I}. \tag{15}$$

Under the assumption that all a_{ij} are strictly positive, Ethier established that every row of the N-matrix must contain at least one negative element and one element greater than unity. By (14) we know that $N_i \cdot \Theta^i = 1$, so N_i must be nonzero for all i.[25] Furthermore, $N_i \cdot \Theta^k = 0$ for $i \neq k$. But, since every element of Θ^k is positive, there must be at least one negative element in N^k. Finally, since $N_i \cdot \Theta^i = 1$ and every N_i contains a negative element, it must also contain an element greater than one, since Θ is column stochastic.[26] Thus, under Ethier's assumptions, every good is an unambiguous friend to at least one factor and an unambiguous enemy to at least one factor.

Now consider the case in which $m > n$ and we require only that production of every good use

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[25] Note that we have used N_i to refer to the i'th row of N and Θ^i to refer to the i'th column of Θ.

[26] The key here is that $N_i \cdot \Theta^i$ is a Θ weighted average, where the fact that Θ is column stochastic means the weights are related by $\sum_{j \in J} \theta_{ij} = 1$. But with one negative element in N_i the remaining terms must sum to >1 with weights that must sum to < 1. Thus, at least one of the elements of N^i must exceed unity.

at least two factors and every factor be used in the production of at least two goods.[27] Unlike the case we have just considered, since **1** is not square, it does not possess an inverse. However, we have already seen that # is square and, under our assumption that $R(A) = n$, does possess an inverse. Thus, we start with the fact that:

$$\mathrm{BB}' = \begin{bmatrix} \Sigma & \Lambda \\ \Theta' & 0 \end{bmatrix} \begin{bmatrix} M & N' \\ Q & R \end{bmatrix} = \begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix}. \qquad \textbf{(16)}$$

Using standard properties of partitioned matrices, we can multiply the second row of the first matrix in (15) by the second column of the second matrix to get

$$\underset{n\times m}{\Theta'} \bullet \underset{m\times n}{N'} = \underset{n\times n}{\mathrm{I}} = \underset{n\times m}{N} \bullet \underset{m\times n}{\Theta}. \qquad \textbf{(17)}$$

The last equality follows from the fact that, if A and B are rectangular matrices, ANBN = (BA)N, and that 3N = 3.[28] We can use Ethier's logic, and our assumption that every factor is used in at least two sectors, to demonstrate that every row of N has at least one negative element.[29]

Unfortunately, we can no longer show that every row of N has an element greater than unity. Jones and Scheinkman (pg. 930) show that Ethier's crucial assumption is that the A-matrix is strictly positive. Once again, N_iA**1**^{i} = 1, but now that there may be zeros in **1** (reflecting zeros in the A-matrix), so the fact that **1** is column stochastic only ensures that *either* the jth row of N has an element

[27]This analysis is drawn from the appendix of Jones and Scheinkman (1977).

[28]The result on multiplication of transposes can be found in Theorem 8.1 of Simon and Blume (1994).

[29]Since Jones and Scheinkman are dealing with the form in the first equality, they prove the result on *columns* of the NN matrix.

exceeding unity *or* the factor whose wage falls when p_j rises (and at least one factor *must* lose) is not used in the production of the jth commodity.

The columns of the N-matrix also provide useful information from the perspective of political-economy modeling. A column of N, say the i'th, shows the proportional impact on the return to factor i as each of the commodity prices is raised by one percent.[30] That is, rows tell us whether or not *some factor* would gain from a change in a particular price, columns that tell us whether or not a *particular factor* would gain from a change in some price. Thus, if we want to know whether every factor has at least one unambiguous friend and at least one unambiguous enemy, we are really asking whether every column of N possesses at least one element greater than one and at least one negative element. To apply the method we have just developed, we would try to determine the contents of the columns of N from the system:

$$\mathrm{B}'\mathrm{B} = \begin{bmatrix} M & N' \\ Q & R \end{bmatrix} \begin{bmatrix} \Sigma & \Lambda \\ \Theta' & 0 \end{bmatrix} = \begin{bmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{bmatrix}. \tag{17}$$

But note that, in the $m > n$ case, we have *M*@F + *N*N@1N = 1. However, in the $m = n$ case, the M-matrix is the zero-matrix (the local FPE result). For this case, Jones and Scheinkman are able to prove the very interesting result that, while every factor has at least one unambiguous enemy, it is not necessarily the case that every factor has an unambiguous friend.

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[30] It is also useful to note that we cannot dispense with the assumption that every factor is used in at least two sectors. For example, Jones and Scheinkman show that in the specific-factors model N possesses one column made up entirely of positive fractions--the column giving the effect of price rises on the return to labor. Recall that in the specific-factors model, every price rise leads to a rise in the wage (though by less than the price rise).

$$\sum_{j \in J} q_{ij} \geq 1.$$

Therefore, we now assume that $m = n$, so from (17) we have:

$$N\mathrm{N}\ 1\mathrm{N} = 3 = 1N.$$

First, we show that every column must contain at least one negative element. Consider the *i*th column of N. Since every row of 1 contains at least two positive elements and $1_i$$N^k = 0$, œ i ... k, every column of N must contain at least one negative element. Thus, there must be at least one commodity such that an increase in its price will cause a reduction in the return to factor *i*. That is, *there is an unambiguous enemy for every factor*.

Now we want to see that every factor need not have an unambiguous friend. While it is true that $1_i$$N^i = 1$, this cannot be used to prove that N^i must contain some element that exceeds unity. Since the elements of the *i*'th *row* of 1 give the distributive share of the *i*'th factor in every sector, there is no reason for them to sum to unity. In fact, that sum can be greater than, less than, or equal to unity. Since that sum gives us some idea of the significance of a given factor in the production of the economy, we can follow Jones and Scheinkman, and define factor *i* to be *economically important*, if
If factor *i* is an economically important factor under this definition, we cannot show that it has an unambiguous friend (though the proof that it has an unambiguous enemy remains valid). Recall the essential role played by the fact that 1 is column stochastic in the demonstration that every row of N has an element greater than unity (see footnote 25). But, if the factor is *unimportant enough* (in the sense that the column sum is less than unity), then at least one element in the *i*th column of N must exceed unity. This is not very good news from the perspective of building simple models of political

economy since, at least under some models, we would expect size ("economic importance" in the sense used here) to play a role in political success. Furthermore, demand for protection, which is empirically much more important to the protectionist process than opposition to protection, is presumed to come from factor-owners that would benefit from protection. Thus, we would like clear results for economically important factors.[31]

One might wonder whether, even for an economically important factor, there must be some package of price increases that results in an unambiguous increase in its real wage. Jones (1985) proves that the answer is yes.[32] We already know that, if $m = n$ and $R(A) = n$:

$$\hat{\mathbf{w}}\Theta = \hat{\mathbf{p}} \Rightarrow \hat{\mathbf{w}} = \hat{\mathbf{p}}N \left[\text{where } N = \Theta^{-1}\right].$$

Thus, the general solution for any factor price change is a weighted average of all commodity price changes

$$\hat{w}_i = \sum_{j \in J} n_{ji} \hat{p}_j .$$

We also know that, since **1** is column stochastic, $\sum_{j \in J} n_{ji} = 1$ (i.e. N has unit column sums). Finally, we know that, for the $m = n$ case, every column must contain at least one negative element. That is, at least one n_{ji} 0 N^i must be negative. Now let the K be the proper subset of commodities k for which n_{ji} > 0 for the given factor i, let $\hat{p}_k = \hat{p} > 0 \ \forall k \in K \subset J$, and let $\hat{p}_j = 0 \ \forall j \notin K.$ Then

[31]Of course, in other types of models, being economically small might be a good thing. For example, Olsonian collective action models predict greater success for more concentrated interests.

[32]This may sound quite plausible, so it is interesting to note that Jones and Scheinkman had conjectured that there need not be any such combination.

$$\hat{w}_i = \sum_{k\in K} n_{ki}\hat{p} = \hat{p}\sum_{k\in K} n_{ki}. \qquad \textbf{(18)}$$

This establishes the result since the existence of a negative element in N^i together with $\sum_{j\in J} n_{ji} = 1$, implies that $\sum_{k\in K} n_{ki} > 1$, so $\hat{w}_i > \hat{p}$. Thus, such a package of relative commodity price changes (i.e. a uniform increase in the prices of the goods in *K* relative to those not in *K*) must unambiguously raise the real reward to any factor *i* in the *m* = *n* case. Note that this will be true even if the relevant factor is large in the sense that no *single* commodity price increase will necessarily serve to raise its real return in a magnified fashion. Also note that while raising the price of all *K*-goods by one percent is sufficient to raise the return to factor *i* by more than one percent, it is entirely possible that it is only necessary to raise the prices of some subset of *K* (possibly even a single price) to accomplish this goal. Finally, it is worth noting that this result, like the others we have discussed in this section is local in the sense that, for any given factor *i*, the members of *K* will depend on the equilibrium.

One should draw two conclusions from this section. First, the general logic of the Stolper-Samuelson theorem is remarkably robust to increases in dimensionality. On the other hand, globally unambiguous friends and enemies are virtually exclusively properties of the 2 × 2 model. Together these reinforce the understanding of the role of formal modeling developed in this monograph. The role of simple models is to focus on some particular set of causes and effects in an analytical environment that abstracts from other factors. In general, we would expect agents to be aware of the price changes that are most likely to affect their particular factor portfolio (i.e. they know the the contents N^i for factors that make up a major part of their income, or at least they know the contents of *K* for those factors). It is this that provides the warrant for political-economy modeling of the sort developed in the next several chapters. However, there is no warrant in the dimensional generalizations of the Stolper-Samuelson theorem for broad cross-national or historical analyses.[33]

Generalizing the Ricardo-Viner Model

The results of the previous section are, in some sense, fairly negative: the more general the model, the weaker are the comparative static results available for use in our political-economy modeling. Since political-economy modeling will involve introducing more complexity into the overall model, this means, fairly obviously, that we will need to make some fairly strong simplifying assumptions. One way of doing this is to assume specific functional forms for all of the relevant relationships, values for all magnitudes, and simulate. To date, this is very rare, and we will not pursue it here. The other approach is to retain some more generality in functional forms but constrain the model structure. The previous chapter developed the 2 × 2 and 3 × 2 models that will serve as the main platforms for analysis is the remainder of this monograph. In this section, we briefly discuss generalizations of the Ricardo-Viner model.

The most straightforward generalization of the 3-factor × 2-good model is simply the (m + 1) ×

[33]See Rogowski (1989, chapter 1) for a particularly confused attempt to do just this. The point is not that one cannot tell a very interesting historical story about factor-based conflict. One can, and Rogowski's story is quite interesting. It just doesn't have anything to do with the Stolper-Samuelson theorem. The point is that, especially if one wants to apply a structure of dimensionality beyond 2 × 2, one cannot claim a connection to Stolper-Samuelson without explicitly justifying the presence of the sort of structure necessary to produce globally unambiguous friends and enemies. See Hall and Nelson (1991) for a detailed discussion.

m specific-factors model.[34] That is, every sector produces with two factors of production, a sector-specific factor (z_j) and a mobile factor (L_j). The great analytical advantage of this generalization is that it behaves virtually identically to the two-sector version of the model. The reason is that, for a small open economy, all general equilibrium effects work through the market for the mobile factor. Of course, this is true in the general neoclassical model, but with many mobile factors, the interdependencies among sectors are considerably more complex.

In the second appendix to the previous chapter we showed that the effect of price changes on the return to the mobile factor can be given by

$$\hat{w} = \sum_{j \in J} b_j \hat{p}_j \text{ where } b_j = \frac{l_{Lj} \dfrac{s_j}{q_{jj}}}{\sum_{k \in J} l_{Lk} \dfrac{s_k}{q_{kk}}} \tag{19}$$

Using this notation, the dimensional generalization of the expression for the effect of a change in the price vector on return to the specific-factor in sector *j* is:

$$\hat{r}_j = \left[b_j + \frac{1}{q_{jj}} \sum_{k \neq j \in J} b_k \right] \hat{p}_j - \frac{q_{Lj}}{q_{jj}} \sum_{k \neq j \in J} b_k \hat{p}_k . \tag{20}$$

These expressions are identical to those in the 3-factor × 2-good case developed in appendix 2 to the previous chapter. With these expressions we can see that the *N*-matrix takes a very specific form.[35]

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[34]The fundamental papers here are Jones (1975) and Ruffin and Jones (1977), though Mussa (1974) presents a brief analysis of this case in an appendix.

[35]Recall that the *N* matrix is the inverse of the 1 matrix and gives elasticities of factor-return with respect to commodity price change.

Letting $\#J = m$ we have:

$$\begin{bmatrix} b_1 + \frac{1}{q_{11}} \sum_{k \neq 1 \in J} b_k & -\frac{q_{L1}}{q_{11}} b_2 & \cdots & -\frac{q_{L1}}{q_{11}} b_m & b_1 \\ -\frac{q_{L2}}{q_{22}} b_1 & b_1 + \frac{1}{q_{11}} \sum_{k \neq 2 \in J} b_k & \cdots & -\frac{q_{L2}}{q_{22}} b_m & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\frac{q_{Lm}}{q_{mm}} b_2 & -\frac{q_{Lm}}{q_{mm}} b_2 & \cdots & b_m + \frac{1}{q_{mm}} \sum_{k \neq m \in J} b_k & b_m \end{bmatrix} \quad \textbf{(21)}$$

Furthermore, the sign pattern is clear:

$$\begin{bmatrix} + & - & \cdots & - & + \\ - & + & \cdots & - & + \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ - & - & \cdots & + & + \end{bmatrix}.$$

From definition of $\$_j$ given in equation (19), and that fact that all sectors must use the mobile factor to produce positive output, we know that the $\$_j$ are all strictly positive and $\sum_{j \in J} b_j = 1$. Thus, with a single commodity-price change, the weighted average property for that price $\left(q_{jj}\hat{r}_j + q_{Lj}\hat{w} = \hat{p}_j\right)$ implies that

$$\hat{r}_j > \hat{p}_j > \hat{w}.$$

so we know that the diagonal elements of N are greater than unity. Given that $n_{jk} < 0$ for $k \ldots j$ or L we must have

$$\hat{r}_j > \hat{p}_j > \hat{w} > \hat{p}_{-j}(=0) > \hat{r}_{-j}. \quad \textbf{(22)}$$

This is essentially identical to the result in the 3-factor × 2-good specific-factor model. Note, in particular, that the return to labor is still characterized by Ruffin and Jones' (1977) neoclassical ambiguity. That is, with the increase in the wage between those of p_j and p_{-j}, the effect of the price change on the real return to labor is ambiguous.[36] However, except for labor, every factor has one (globally) unambiguous friend, while all other commodities are (globally) unambiguous enemies. Similarly, every commodity is an unambiguous friend to exactly one factor, and an enemy to all other factors. It should be clear that this will dramatically simplify political-economic analysis.

With $m > 2$ goods, of course, it is possible that multiple prices can be increased relative to some numeraire (say, by a tariff). Using the system $\hat{\mathbf{w}} = \hat{\mathbf{p}}'N$, there is no particular difficulty in analyzing the case in which the vector $\hat{\mathbf{p}}$ has multiple positive elements. We already know, from equations (19) and (20), what the expressions for the $\hat{w}$ and $\hat{r}_j$ look like. First, suppose that good 1 is one of several commodities that experience an increase in price. From equation (20) it is easy to see that, while the coefficient of $\hat{p}_1$ exceeds unity, any other price rise reduces $\hat{r}_i$. The reason is that other sectors experiencing price rises also seek to expand by hiring more labor, thus bidding up the wage.

From the perspective of a formal theory of political-economy, we would like to know the relative impact of a package of commodity price changes (say, a package of tariff increases) on the returns to various factors. To get some idea of the interactions involved here, we can focus on goods 1 and 2. Jones (1975) develops a clear analysis based on the following expression derived using

[36]The purpose of the Ruffin and Jones (1977) paper is to argue, from the fact that commodity j is assumed to be an import (since the price increase is induced by an increase in the tariff), that there is a presumption that the effect on the real wage is negative.

equation (20):

$$\hat{r}_1 - \hat{r}_2 = \left[\frac{1}{q_{22}} b_1 + \frac{1}{q_{11}} \sum_{k \neq 1 \in J} b_k \right] \left(\hat{p}_1 - \hat{p}_2 \right) + \frac{\left(q_{L1} - q_{L2} \right)}{q_{11} q_{22}} \sum_{j=3}^{m} b_j \left(\hat{p}_2 - \hat{p}_1 \right) \qquad \textbf{(23)}$$

First, note that the coefficient of $\left(\hat{p}_1 - \hat{p}_2 \right)$ is identical to the coefficient of $\hat{p}_1$ in equation (20) except for the $\frac{1}{q_{22}}$ multiplying $\$_1$. But since 2_{22} is a positive fraction, $\frac{1}{q_{22}} > 1$, and the first term in square brackets in equation (23) is also positive and greater than unity. Thus, as we already know from the analysis that yielded inequalities (22), if p_1 is raised and all other commodity prices are held constant, the proportional increase in r_1 relative to r_2 must be greater than the proportional increase in p_1 relative to p_2.[37] Now suppose that $\hat{p}_1 = \hat{p}_2 > \hat{p}_j$ (j … 1 or 2), so the full effect of the change in the commodity price vector on $\hat{r}_1$ relative to $\hat{r}_2$ works through the second term on the right hand side of equation (23). In this case $\hat{r}_1 > \hat{r}_2$ if the distributive share of labor is greater in industry 1 than in industry 2. This suggests, for example, that if there were two import-competing industries contemplating lobbying for a uniform proportional increase in the tariff of one percent, abstracting from strategic considerations, the willingness to pay for such a change would generally differ between industries even though the proportional increase in the price was identical.

Although the specific-factors model is very simple, it remains a complete general equilibrium model in the sense that it captures the essential reality of intersectoral and inter-factoral interdependence. Even a single commodity-price change has economy-wide implications for output and for the level and distribution of household income. These are essential building-blocks of any

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[37]Since this applies to any other factor j … 1 or L, we can use this in a more formal demonstration of the result in (22).

theory of political-economy. Sometimes, however, when linked to an explicit political model, even this very well-behaved complexity can result in comparative static results which are indeterminate.[38] One not uncommon response to this problem is to "switch off" these general equilibrium linkages by assuming the existence of a Ricardian sector whose price is not an object of policy.[39] For the small open economy, the existence of such a sector locks in the nominal wage, thus eliminating all supply-side interdependencies among sectors (except, of course, between a given sector and the Ricardian sector). Because the Ricardian sector is not an object of policy, this structure reduces the production-side of the model to a collection of essentially unconnected sectoral partial equilibria. When used in political-economy modeling, this structure has the effect of eliminating all interdependence between sectors except that directly implied by political interaction over the determination of some policy. While this provides a convenient framework for political-economy modeling, one should be sceptical of claims that it is a general equilibrium framework.

An alternative generalization of the specific-factors model, more in the spirit of the generalizations of the HOS model we have already considered, retains the assumption that production in each sector involves the use of one specific factor, but permits multiple mobile factors. Ferguson (1982) considers the case of V mobile factors and m industries, each of which also uses a single

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[38]That is, under the assumptions of the model, we cannot determine the sign of a particular comparative static effect.

[39]See, for example, Rodrik (1986), Grossman and Helpman (1994), or Riezman and Wilson (1995). Note, by "Ricardian" we mean a sector that produces under constant returns to scale using only the general purpose factor (Labor) as an input.

specific factor.[40] The production functions are assumed to be characterized by constant returns to scale (i.e. homogeneity of degree 1) in the specific and mobile factors taken together, but the structure of the relationship between specific and mobile factors is left open. There are now $\#\boldsymbol{v} = V$ full-employment conditions for the mobile factors

$$A\mathbf{y} = \bar{\mathbf{v}}; \tag{24}$$

$\#\boldsymbol{t} = T$ full-employment conditions for the specific-factors

$$\tilde{A}\mathbf{y} = \bar{\mathbf{t}}; \tag{25}$$

and m zero-profit conditions

$$A'\mathbf{w} + \tilde{A}'\mathbf{r} = \mathbf{p}. \tag{26}$$

where A is a $V \times m$ matrix of technical coefficients for the mobile factors, $\tilde{A}$ is a $Z \times m$ (diagonal) matrix of technical coefficients for the specific-factors, and the prime denotes transposition.

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[40]Berglas and Razin (1974) and Thompson (1989) analyze related lower-dimensional models. Berglas and Razin present a brief analysis of the 2-good case in which production in each sector requires the use of one specific-factor and two mobile factors, first under linear homogeneity in all factors and and separability between the mobile and specific factors, and then under more general conditions. Not surprisingly, in the first case they find that increasing the price of one good relative to the other raises the real return to the specific factor, and raises the nominal return to the intensively used mobile factor (but not necessarily its real return), while the latter case is more complex. Thompson presents a more detailed analysis of a closely related model in which only one sector uses a specific factor with two mobile factors, while the other sector uses only the two mobile factors. Although he is able to use the lower dimensionality to good effect with respect to the particular issue with which he is concerned, Thompson's methodology is essentially the same as that used by Ferguson.

Differentiation and manipulation of this system following that in equations (3) - (7) above yields

$$\begin{bmatrix} S_{vv} & S_{vt} & A \\ S_{tv} & S_{tt} & \tilde{A} \\ A' & \tilde{A}' & 0 \end{bmatrix} \begin{bmatrix} d\mathbf{w} \\ d\mathbf{r} \\ d\mathbf{y} \end{bmatrix} = \begin{bmatrix} d\mathbf{z} \\ d\mathbf{t} \\ d\mathbf{p} \end{bmatrix}. \tag{27}$$

Referring back to equation (8), it should be clear that equation (26) is just a special case of equation (8) for which we have detailed knowledge of the last V rows of the overall technology matrix (i.e. the rows that make up $\tilde{A}$).[41] Since $I = V + T > m$, the question is whether the additional structure allows us to say more than we were able to say in the general $m > n$ case. The answer is going to depend on whether that structure allows us to say more about the economy-wide substitution matrix (S), which now is made up of four distinct submatrices: S_{vv} and S_{tt} refer to the effects of changes in $\boldsymbol{w}$ and $\boldsymbol{r}$ on the economy-wide usage of $\boldsymbol{z}$ and $\boldsymbol{t}$ respectively, while $S_{vt} = S_{tv}'$ gives the effects of $\boldsymbol{w}$ and $\boldsymbol{r}$ on economy-wide usage of $\boldsymbol{t}$ and $\boldsymbol{z}$.

Following the methodology developed by Diewert and Woodland (1977) and Chang (1979), Ferguson uses the envelope results on the national product function to derive constraints on the inverse matrix that yield results on the link between commodity-price changes and factor-price changes. Ferguson's main conclusion is that, without additional structure on production functions and the economywide relationship between specific and non-specific factors, the existence of specific-factors is not much help. However, if we are willing to assume (in addition to linear homogeneity in all inputs)

[41]Of course, while it remains true that production of every commodity uses at leat two factors of production (in fact Ferguson assumes that production of every good uses strictly positive quantities of every mobile factor), it is not the case that every factor is used in at least two sectors. Rather, as in any specific-factors model, specific-factors are used in only one sector.

separability between specific and non-specific factors in the production functions, and aggregate complementarity between specific and non-specific factors, then we can recover results like those in the 3-factor × 2-good and (m + 1)-factor × m-good specific-factor cases.[42] While these are quite strong additional assumptions on production, Ferguson demonstrates that they can be satisfied.

Unsurprisingly, the main message from generalizations of the specfic-factors models is very much like that from generalizations of the neoclassical model (of which it is a special case): unless our generalizations are, themselves, very highly structured, we will find it hard to get strong comparative static results. Again, this does not mean that the results from the simple models are uninformative with respect to the main channels through which economic effects are propagated. It does mean that these models are far from complete representations of the world as we generally observe it; and, as a result, it does mean that the stories these theorems frame about the world must be told in a manner consistent with the assumption structure of the theorem.

Other Generalizations of the Neoclassical Production Model

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[42]Separability of a production function means that we can partition the the input vector into n exhaustive, mutually exclusive subsets such that a change in the use of an element in one subset has no effect on the marginal rate of technical substitution among elements of another subset. Thus, the production function can be written as a top level aggregator of n lower-level functions that can each be optimized separately. In this case, that means that a change in the level of the specific-factor has no effect on the marginal rates of technical substitution among mobile factors. That is, the slope of the isoquant between two mobile factors is unaffected by the level of input of the specific-factor. Since we have already assumed that the production functions are homogeneous of degree one in $\mathbf{z}_j$, separability implies that the lower-level aggregators are homothetic. As a result, the production functions are characterized by homothetic separability, which means that the cost function is also separable in the same partition. See Blackorby, Primont and Russell (1978) for detailed analysis of many aspects of separability and functional structure.

To this point we have considered only models that are based on a very simple version of the traditional neoclassical production model. Trade economists have always been aware that the reality of economic life is considerably more complex than that reflected in these basic models. As a result, there is a substantial literature that attempts to determine the effects on the main comparative static results from relaxing the production assumptions of the basic model.[43] In this section we briefly discuss three such generalizations (intermediate goods, joint products, and variable returns to scale) and their implications for political economy modeling--especially results of the Stolper-Samuelson sort. As with the previous sections of this chapter, the major reason for such a review is to remind ourselves that the models we will be dealing with are considerable simplifications.

...More Here...

Concluding Remarks

The Stolper-Samuelson theorem is a theorem. That is, it is a *mathematical* fact, a logically true proposition derived from the assumptions of the Heckscher-Ohlin-Samuelson model. By revealing the importance of indirect effects of policies, the Stolper-Samuelson theorem has played an important role in the analysis of both trade policy and the political economy of trade policy. But it is very important to be clear that the Stolper-Samuelson theorem is neither a general property of neoclassical

[43]There are also generalizations that relax the institutional assumptions (especially those asserting the existence of complete and perfect markets). We will return to these issues when we discuss unemployment. At least since the foundational work on trade and trade policy under distortions by Haberler, Johnson, and Bhagwati, international economists have studied the implications for policy of relaxing the technological and institutional assumptions of the basic model (sometimes exhausting) detail. We will discuss this body of research in chapter 7.

models, nor an empirical fact about the world we live in.[44]

[44] A detailed discussion of the empirical status of the Stolper-Samuelson theorem is well beyond the confines of this monograph. We have argued that the HOS theory is a useful simplification of a (much!) more complex reality, and that it constitutes a particularly useful foundation for political-economic analysis. Interestingly, this remains true even though virtually every attempt to test its predictions has been resoundingly falsified. See Leamer and Levinsohn (1995) for an insightful review of this literature. As Leamer and Levinsohn are at some pains to point out, falsification of such a simple theory is not tantamount to the claim that it is without empirical content or without value, only that the world is far too complex to make "testing" a sensible strategy. The current boom in research that seeks to use the Stolper-Samuelson theorem as the basis for empirical work on the effect of trade on wages is controversial precisely because of the fragility of the strong predictions.

Appendix I

Linear Algebra[45]

1. **Some terminology from linear algebra**.

Linear algebra plays a major role in the analysis of this chapter. As a result, this appendix attempts to provide an introduction to some of the language and intuition of linear algebra. We will assume that the basic notation and operations of matrix algebra are familiar. In this section we simply review some terminology.

Suppose we have a pair of linear equations in two unknowns:

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 &= y_1 \\ a_{21} x_1 + a_{22} x_2 &= y_2 . \end{aligned} \tag{1}$$

The unknowns are x_1 and x_2, the variables y_1 and y_2 are exogenous variables, and the a_{ij}'s are parameters that link the x_j's to the y_i's. In parametric form, each of these equations is given by a slope $\left(\frac{a_{i2}}{a_{i1}} \right)$ and an intercept (y_i). The slope defines an infinite family of parallel lines. In higher dimensions it is easy to work with the *normal vector*, that is the vector perpendicular to the family of hyperplanes, given by $\{a_{i1}, a_{i2}, \ldots , a_{in}\}$.

We can represent a system of linear equations in matrix form as

---

[45] There are many textbook treatments of linear algebra. Parts II and VI of Simon and Blume (1994) provide an excellent introduction. A more condensed presentation can be found in Chapter 2 of Gale (1960).

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \textbf{(2)}$$

This can be collapsed to

$$A\boldsymbol{x} = \boldsymbol{y} \,. \quad \textbf{(3)}$$

where A is a matrix, and $\boldsymbol{x}$ and $\boldsymbol{y}$ are vectors. We can also think of this as a function f: ú^2 6 ú^2, where $f(\boldsymbol{x}) = A\boldsymbol{x}$ is a *linear* function. If $f(\boldsymbol{x}) = A\boldsymbol{x} + \mathbf{b}$, where $\mathbf{b}$ is an m-vector of constants, we refer to this as an *affine* function. If $\boldsymbol{y} = \mathbf{0}$ (the zero vector) equation (A1.3) is called *homogeneous*.

2. **Rank, Dimension, and Solution of Linear Systems**

In this appendix we will briefly review the notion of the *rank* of a system of linear equations and its relationship to solutions of such a system.

3. Lemma 1 (Chang,1979): Within the framework of the model, B is nonsingular if and only if the rank of A, $R(A)$, is n.

Proof: It is obvious that det(B) ... 0 implies $R(A) = n$. Since uN$Su < 0$ for all nonzero u nonproportional to w [by (11)], the quadratic form satisfying AN$u = 0$ is also negative definite for all nonzero u not proportional to w. If $u = t\boldsymbol{w}$, for some scalar t, u ... 0, $\boldsymbol{w} > 0$, then AN$u = tA$N$\boldsymbol{w} = t\boldsymbol{c}$ ... 0, since t ... 0 and $\boldsymbol{c} > 0$. Thus, any nonzero $\boldsymbol{u}$ satisfying the constraints AN$\boldsymbol{u} = 0$ implies that $\boldsymbol{u}$ is not proportional to $\boldsymbol{w}$. Thus, uNSu is negative definite subject to AN$\boldsymbol{u} = 0$, $\boldsymbol{u}$ ... 0. It follows that det(B) has the sign $(-1)^r$ ... 0. €

Appendix II:

The Method of Comparative Statics

Comparative statics is a method of analysing the impact of a change in the parameters of a model by comparing the equilibrium that results from the change with the original equilibrium. The previous chapter has already introduced the basic idea of comparative static analysis in an informal fashion. There we asked: what is the effect of a change in prices, taken to be parametric for the small country, on the equilibrium real factor returns. The answer to that question is given in the Stolper-Samuelson theorem. Assuming 2 goods and 2 factors of production, along with the other assumptions of the model, we were able to develop simple comparative statics in an essentially informal, graphical fashion. As we move into a more complex analytical environment, it will prove useful to be a bit more systematic in describing the tools used in the method.

The style of reasoning that is formalized in the method of comparative statics is as old as economics, however, the modern approach to comparative statics (in economics) was formalized by Hicks (1939) and, especially, Samuelson (1947). Once we have specified the equilibrium relationships among the exogenous and endogenous variables that describe the system we seek to analyze, we perturb the system by changing one (or more) of the exogenous variables and use the assumed structure of the system to trace through the effect of such changes on the equilibrium values of the endogenous variables. For small changes, this method is formalized as taking total differentials of the equilibrium equations with respect to the relevant exogenous variables and solving for the changes in the endogenous variables.

Suppose that we can represent the equilibrium of some system by a set of equations involving

M endogenous variables $\boldsymbol{w} = \{w_1, \ldots , w_M\}$ and N exogenous variables $\boldsymbol{p} = \{p_1, \ldots , p_N\}$, of the form:

$$
\begin{aligned}
c^1(\boldsymbol{w},\boldsymbol{p}) &= 0 \\
&\vdots \\
c^M(\boldsymbol{w},\boldsymbol{p}) &= 0
\end{aligned}
$$

We can write this system more compactly in matrix form as

$$C(\boldsymbol{w};\boldsymbol{p}) = 0,$$

where $C(\cdot)$: $W \times P$ 6 ú ^{M}, W d ú ^{M} and P d ú ^{N}. That is, the matrix $C(\cdot)$ defines a relationship in an $M \times N$-dimensional real space between the N-dimensional vectors of exogenous variables and the M-dimensional vectors of endogenous variables. Suppose we assume that $M = N$ so that the number of equilibrium conditions is equal to the number of endogenous variables.

We want to use calculus methods to study the effect of a small change in the parameters of the system on the equilibrium values of the endogenous variables. To aid us in this effort, we will take advantage of an important result caled the *implicit function theorem* which says that if our initial point satisfies the equations $C(\cdot) = 0$; if the functions $c^i(\cdot)$ are differentiable; and if the determinant of the $M \times M$ (Jacobian) matrix $\left[\frac{\partial c^j(\boldsymbol{w},\boldsymbol{p})}{\partial w_i}\right]$ is nonzero, then in the neighborhood of the initial equilibrium we can write the endogenous variables of the system as differentiable functions of the exogenous variable, i.e.: $w_j = \mathrm{F}(\boldsymbol{p})$.[46] Thus, suppose that $(\boldsymbol{w}°, \boldsymbol{p}°)$ is a solution to this system of equations, that the function $C(\cdot)$ is continuously differentiable, and that Jacobian matrix is

[46]The implicit function theorem is covered in any calculus text. Particularly clear discussions with application to comparative statics of economics models can be found in Silberberg (1990, chapter 5) and Novshek (1993, chapters 6 and 7).

$$
DC_w(\boldsymbol{w}^\circ,\boldsymbol{p}^\circ) = \begin{bmatrix} \frac{\partial c^1(\boldsymbol{w}^\circ,\boldsymbol{p}^\circ)}{\partial w_1} & \cdots & \frac{\partial c^1(\boldsymbol{w}^\circ,\boldsymbol{p}^\circ)}{\partial w_M} \\ \vdots & \ddots & \vdots \\ \frac{\partial c^N(\boldsymbol{w}^\circ,\boldsymbol{p}^\circ)}{\partial w_1} & \cdots & \frac{\partial c^N(\boldsymbol{w}^\circ,\boldsymbol{p}^\circ)}{\partial w_M} \end{bmatrix} \neq 0.
$$

By the assumptions that we have just made, this matrix is invertible.[47] To calculate the impact of small changes in $\boldsymbol{p}$ on $\boldsymbol{w}$, differentiate with respect to $\boldsymbol{p}$ using the chain rule

$$
\begin{aligned}
Dc_w[\boldsymbol{w}(\boldsymbol{p}),\boldsymbol{p}]D\boldsymbol{w}(\boldsymbol{p}) + Dc_p[\boldsymbol{w}(\boldsymbol{p}),\boldsymbol{p}] &= 0 \\
Dc_w(\boldsymbol{w}^\circ,\boldsymbol{p}^\circ)D\boldsymbol{w}(\boldsymbol{p}^\circ) + Dc_p(\boldsymbol{w}^\circ,\boldsymbol{p}^\circ) &= 0 \\
D\boldsymbol{w}(\boldsymbol{p}) &= -Dc_w(\boldsymbol{w}^\circ,\boldsymbol{p}^\circ)^{-1}Dc_p(\boldsymbol{w}^\circ,\boldsymbol{p}^\circ)
\end{aligned}
$$

The comparative static results are contained in the elements of the $M \times N$ matrix

[47] The *inverse function theorem* asserts that a continuously differentiable function, like $C(\cdot)$, is invertible in a small neighborhood of the equilibrium $\boldsymbol{w}^\circ$ if the Jacobian matrix is nonsingular. This implies that $\boldsymbol{w}^\circ = C^{-1}(0, \boldsymbol{p}^\circ)$ is the locally unique solution to the equilibrium conditions. That is, the function is *locally univalent*. Furthermore, under the same assumptions, the inverse function theorem also says that the locally unique vector $\boldsymbol{w}$ that satisfies $C(\boldsymbol{w}, \boldsymbol{p}) = 0$ varies continuously with $\boldsymbol{p}$ near $\boldsymbol{p}^\circ$. That is, there is a continuous function, $w(\boldsymbol{p})$ such that

$$C(\boldsymbol{w}(\boldsymbol{p}),\boldsymbol{p}) = 0.$$

$$Dw(p) = \begin{bmatrix} \frac{\partial w^1(p^\circ)}{\partial p_1} & \cdots & \frac{\partial w^1(p^\circ)}{\partial p_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial w^M(p^\circ)}{\partial p_1} & \cdots & \frac{\partial w^M(p^\circ)}{\partial p_N} \end{bmatrix}$$

For example, if parameter p_1 changes from p_1° to p_1N, the equilibrium value of, say, the second endogenous variable, w_2, changes from w_2° to approximately

$$w_2' \approx w_2^\circ + \frac{\partial w_2(w^\circ)}{\partial p_1}(p_1' - p_1^\circ).$$

This is just a linear approximation of the true value, which is accurate only for small (actually only for infinitesimal) changes.

Appendix III

Properties of the Unit Cost Function's Hessian Matrix

We have assumed that the production functions, $y_j = f^j(z)$, are positive, homogeneous of degree 1, twice smoothly differentiable, and strictly concave for $z > 0$. Given these assumptions, we can represent the technology of sector j with a *unit cost function*:

$$c^j(\mathbf{w}) := \min_{\mathbf{z}} \left\{ \mathbf{w} \bullet \mathbf{z} \middle| f^j(\mathbf{z}) \geq 1, \mathbf{z} \geq 0 \right\}.$$

We note in the text that, under these assumptions, the unit cost function is positive, linearly homogeneous, twice differentiable, and strictly concave for $\boldsymbol{w} \geq 0$.

Now we want to show sketch demonstrations of several useful results on the $m \times m$ Hessian matrix of the unit cost function– c^j_{ww}. First, this matrix is *symmetric*. Young's theorem states that, for a function f: $\Re^m \to \Re^1$, if f_i and f_k exist everywhere and f_{ik} and f_{ki} are continuous at the initial equilibrium point, say $z°$, then $f_{ij}(z°) = f_{ki}(z°)$. Since the unit cost functions are twice smoothly differentiable, Young's theorem applies, and the Hessian is symmetric.

Second, we want to show that: $c^j_{ww}\mathbf{w} = 0$. Since $c^j(\boldsymbol{w})$ is homogeneous of degree 1, the $c^j_i(\boldsymbol{w})$ are homogeneous of degree zero; and so, by Euler's theorem for homogeneous functions $c^j_{ww}\mathbf{w} = 0$. It follows from the symmetry of the Hessian that $\mathbf{w}c^j_{ww} = 0$.

Next, we want to demonstrate that the Hessian matrix is negative semidefinite. But Bernstein and Toupin (1962, theorem 1) show that: If f: $\Re^m \to \Re^1$ is strictly concave on its domain, then its Hessian matrix is negative semidefinite everywhere.[48] Since the unit cost function is strictly concave,

[48] Actually, Bernstein and Toupin prove the result for strictly convex functions. We simply make the appropriate adjustment.

this theorem applies, and the Hessian of the unit cost function is negative semi-definite everywhere.

What we want for our next result, however, is that the Hessian be negative definite, i.e. that $\mathbf{u}' c^j_{ww} \mathbf{u} < 0$ for any nonzero $\boldsymbol{u}$ not proportional to $\boldsymbol{w}$.[49] The standard approach in the literature is to assume this. In fact, this is precisely the meaning of "a *regular minimum* in the sense of Samuelson" (1947, pg. 68; 1960, pg. 2). Interestingly, the assumptions that the cost function is strictly concave and twice differentiable ensure that the Hessian matrix is negative definite almost everywhere (Berstein and Toupin, theorem VI), where "almost everywhere" means that the statement is true except on a nowhere dense subset of the cost function's domain. Thus, the assumption seems quite reasonable.

Finally, supposing that production of good j requires the services of m distinct factors of production, and if we denote the rank of the Hessian matrix by $R\left(c^j_{ww}\right)$ we want to show that $R\left(c^j_{ww}\right) = m - 1$. The rank of an $m \times n$ matrix A, is the number of linearly independent columns in A. The *null space* of A, $NS(A)$, is the set of vectors that solve the homogeneous system $A\boldsymbol{x} = 0$, where $\boldsymbol{x}$ is an n-dimensional vector. That is, $NS(A)$ / $\{\boldsymbol{x}: A\boldsymbol{x} = 0\}$. We will denote the *dimension* of the null space by $N(A)$. It is a fundamental result from linear algebra that, $R(A) + N(A) = n$. Thus, if we can show that $R(N) = 1$, since c^j_{ww} is $m \times m$, we will have shown that $R\left(c^j_{ww}\right) = m - 1$. The null space of the Hessian matrix of the unit cost function is defined as

$$NS\left(c^j_{ww}\right) := \left\{\mathbf{u} \,\middle|\, c^j_{ww}\mathbf{u} = 0\right\}.$$

But we already know that $c^j_{ww}\mathbf{w} = 0$, and that $\mathbf{u}' c^j_{ww} \mathbf{u} < 0$ for all $\boldsymbol{u}$ not proportional to $\boldsymbol{w}$, so

$$NS\left(c^j_{ww}\right) = \left\{\mathbf{u} \,\middle|\, \mathbf{u} = t\mathbf{w}, t \in \mathbb{R}\right\}.$$

---

[49] The restriction to $\boldsymbol{u}$ follows from the previous result that $\sum c^j_{ww} w = 0$.

Thus, the dimensionality of the null space is 1 so, since $R\left(c_{ww}^{j}\right)+NS\left(c_{ww}^{j}\right)=m$, we have that $R\left(c_{ww}^{j}\right)=m-1$.

What kind of information is contained in c_{ww}^{j}? Since $\frac{\partial c^{j}\left(\mathbf{w}\right)}{\partial w_{i}}=a_{ij}\left(\mathbf{w}\right)$, by Shepard's lemma, the Hessian provides information about substitution around the industry *j* unit isoquant in response to changes in factor-prices. Using the notational convention that $a_{ij}^{k}:=\frac{\partial a_{ij}\left(\mathbf{w}\right)}{\partial w_{k}}$, it is easy to see that $c_{ik}^{j}=\frac{\partial^{2}c^{j}\left(\mathbf{w}\right)}{\partial w_{k}\partial w_{i}}=\frac{\partial a_{ij}\left(\mathbf{w}\right)}{\partial w_{k}}=a_{ij}^{k}$, so

$$\left[c_{ww}^{j}\left(\mathbf{w}\right)\right]=\begin{bmatrix}\nabla a_{1j}\left(\mathbf{w}\right)'\\ \vdots\\ \nabla a_{mj}\left(\mathbf{w}\right)'\end{bmatrix}$$

where each row of the matrix on the right hand side is given by a transposed gradient of an industry *j* unit-factor demand function. Thus, from the negative definiteness of this matrix, we already know that an increase in a factor's own price must lead to a decrease in the usage of that factor by sector *j* in a unit of output, since the diagonal elements of a negative definite matrix must be negative. More generally, factor *i* and factor *k* are said to be *Allen substitutes* if $\frac{\partial z_{ij}}{\partial w_{k}}>0$, and Allen complements if this derivative is less than zero. But since $a_{ij}\,y_j = z_{ij}$, it should be clear that $\frac{\partial z_{ij}}{\partial w_{k}}=y_{i}\frac{\partial a_{ij}}{\partial w_{k}}$, so $y_{j}c_{ww}^{j}=y_{j}a_{ij}^{w}$ is the substitution matrix for industry *j*.

Suppose that we are interested in the economywide response in the use of factor *i* to a change in the price of factor k. To find this magnitude, we would sum the individual sectoral effects across all *I* sectors. We will denote this *economywide substitution effect* by

$$s_{ik} := \sum_{j \in J} a_{ij}^{k} y_j ,$$

and, as in the main body of the text, we will denote the $m \times m$ matrix of these terms S. Using the properties of the c_{ww}^{j} matrices that we have already developed, and the definition of S, it is straightforward to show that S is symmetric, negative semidefinite, with $S\boldsymbol{w} = \boldsymbol{w}'S = 0$, and $R(S) = I - 1$.