Groups with Intersecting Interests

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April 28, 2005

Abstract

We model a situation involving N groups of players. Each player belongs to a single group, and enjoys a public good that is generated by members of that group. However, members of each group are also, to a greater or lesser extent, advantaged or disadvantaged by the public goods produced by other groups. We provide a simple but rigorous analysis of existence, uniqueness and comparative static properties of Nash equilibrium in such a model.

1 Introduction

Collective action problems often involve many players, each of whom belongs to a single group whose collective interest to some extent either conflicts or coincides with those of other groups. This note suggests and analyzes a simple model of such interdependence. Each player contributes towards a public good that is not only enjoyed by all her fellow group members, but may also be either a good or a bad from the viewpoint of members of all other groups. Our analysis exploits and extends a novel approach developed by Cornes and Hartley [4] in their analysis of the pure public good model. By avoiding the need to analyze mappings in high dimensional spaces, it allows us to model situations involving many groups and players. The individual players may differ with respect to preferences, endowments and also productivities as public good generators, and the groups may be of different sizes and contain heterogeneous members. In short, our analysis does not restrict itself to games involving identical players or identical groups. We establish conditions under which the existence of a unique Nash equilibrium in pure strategies is assured, and also provide a simple graphical treatment of the model and of its comparative static properties.

Existing literature on the collective action problem in its various guises typically either models the free rider problem confronting a single group of individuals with a common interest (Olson [7], Stigler [8]), or, if it explicitly acknowledges the presence of two or more groups, treats the groups themselves as the basic decisionmaking units, as if each were a single optimizer (Grossman and Helpman [6]). Bruce [3] provides an interesting attempt to combine intra- and inter-group considerations. His model involves conflict between an alliance of two members and a single adversary. The original motivation of the present model was to accommodate both features within a single tractable framework that can accommodate many groups, each consisting of many heterogeneous members. Our initial concern was with collective action stories in which the groups have mutually conflicting interests. However, our approach applies equally to situations in which a public good provided by members of one group partially or wholly spills over to the benefit of members of others.

2 The Model

There are n individuals and N groups. Each player belongs to precisely one group. We use the subscript i to refer to player i, and the subscript j to refer to group j. The set of all individuals is I, and the set of individuals in group j is the set I_j. Hence the statement $i \in I_j$ instructs us to confine attention solely to members of group j. The set of all groups is denoted by J. The numbers of groups and players, and the assignment of players to groups, are exogenous. One may think of membership of a group as being determined, for example, by pre-existing ownership of a specific factor, by claims on profits of a particular industry, by geographical location, or by cultural or ethnic characteristics.

The utility function of player i in group j is denoted by $u_i(x_i, \Pi_j)$, where x_i is player i's consumption of a private good and Π_j is the level of public good enjoyed by all members of group j.

The quantity Π_j is determined as

$$\Pi_j = G_j + \theta \sum_{k \in J \diagdown \{j\}} G_k \qquad (1)$$

where

$$G_j = \sum_{i \in I_j} g_i,$$

the statement $k \in J \diagdown \{j\}$ denotes the summation over all groups except for group j, and θ is an exogenous parameter. The quantity $g_i \geq 0$, $i \in I_j$, is player i's contribution to the good G_j.

For reasons that we elucidate below, we will assume that $-\frac{1}{N-1} \leq \theta \leq 1$. Clearly, if $\theta < 0$, extra provision by members of one group generates a negative externality, or bad, for members of all other groups. By contrast, $\theta > 0$ describes a model in which a good provided by members of one group partially or - if $\theta = 1$ - fully spills over to provide benefits for the members of others. Finally, if $\theta = 0$, then the model is one of a set of completely independent 'island economies', each described by the standard pure public good model.

We adopt the following assumptions concerning preferences:

A1: Well-behaved preferences For all i, j, $u_i(x_i, \Pi_j)$ is everywhere strictly increasing and strictly quasiconcave in both arguments. It is also everywhere continuous[1].

A2: Strict Normality For every group, both goods are everywhere strictly normal.

Concerning individual resource constraints, we assume:

A3: Constant costs Every player faces a constant marginal rate of transformation between private good consumption and public good contribution.

Player i's income, m_i, is exogenous for all i. Although [for the moment] every player has constant costs as a contributor to her public good, we generally allow unit costs to differ across players. Player i's unit cost coefficient

[1] Note that we do not require the assumption that $u_i(x_i, \Pi_j)$ is differentiable.

as a contributor to her group's public good is denoted by c_i, and her budget constraint is

$$x_i + c_i g_i \leq m_i. \tag{2}$$

At a simultaneous Nash equilibrium every player chooses a contribution level and private good consumption to maximize utility given the choices made by all other players. Since utility functions are strictly increasing in both arguments whenever both arguments are strictly positive, all players will spend their entire incomes at a Nash equilibrium. We may therefore confine attention to situations in which, for every player, (2) holds with equality.

3 Individual behavior

Consider first the problem facing player $i \in I_j$ when the public good contributions of all other players are given. Our assumptions ensure that, along the locus of utility-maximizing allocations supported by the relative price, c_i, player i's preferred total quantity of the public good may be described by a single-valued function, $D_i(M_i, c_i)$, where M_i is player i's full income: $M_i = m_i + c_i \Pi_{j \backslash i}$ where $\Pi_{j \backslash i}$ denotes the total of all contributions to the public good enjoyed by members of group j except for that of player i. Since c_i is constant by assumption, we can define player i's Engel curve as $\psi_i(M_i) = D_i(M_i, c_i)$. Normality implies that this has an inverse: $M_i = \psi_i^{-1}(\Pi_j)$. This uniquely determines the full income of a utility-maximising player i in group j who chooses to consume the quantity Π_j.This locus is the dashed income expansion path [IEP] in Figure 1.

Following the procedure in Cornes and Hartley, define the following sets:

$$\begin{aligned} \mathbf{A}_i &= \left\{ \Pi_{j \backslash i} \left| \psi_i \left(m_i + c_i \Pi_{j \backslash i} \right) - \Pi_{j \backslash i} \geq 0 \right. \right\} \\ \mathbf{B}_i &= \left\{ \Pi_{j \backslash i} \left| \psi_i \left(m_i + c_i \Pi_{j \backslash i} \right) - \Pi_{j \backslash i} < 0 \right. \right\}. \end{aligned}$$

Normality implies that $c_i \psi_i' \left(m_i + c_i \Pi_{j \backslash i} \right) < 1$, so that $\psi_i \left(m_i + c_i \Pi_{j \backslash i} \right) - \Pi_{j \backslash i}$ is decreasing in $\Pi_{j \backslash i}$ for all $\Pi_{j \backslash i} > 0$. Thus the sets $\mathbf{A}_i$ and $\mathbf{B}_i$ are intervals of the real line, $(-\infty, K_i]$ and (K_i, ∞) respectively. We call the point K_i player i's 'dropout point'[2].

Now define the function $r_i(\Pi_j, m_i, c_i)$ as follows:

$$\text{If } \Pi_{j \backslash i} \in \mathbf{B}_i,\ r_i(\Pi_j, m_i, c_i) = 0$$

[2]We follow existing treatments in supposing that player i's dropout point, K_i, is finite. But our analysis can readily handle the possibility that, for some or all $i \in I$, $K_i = \infty$.

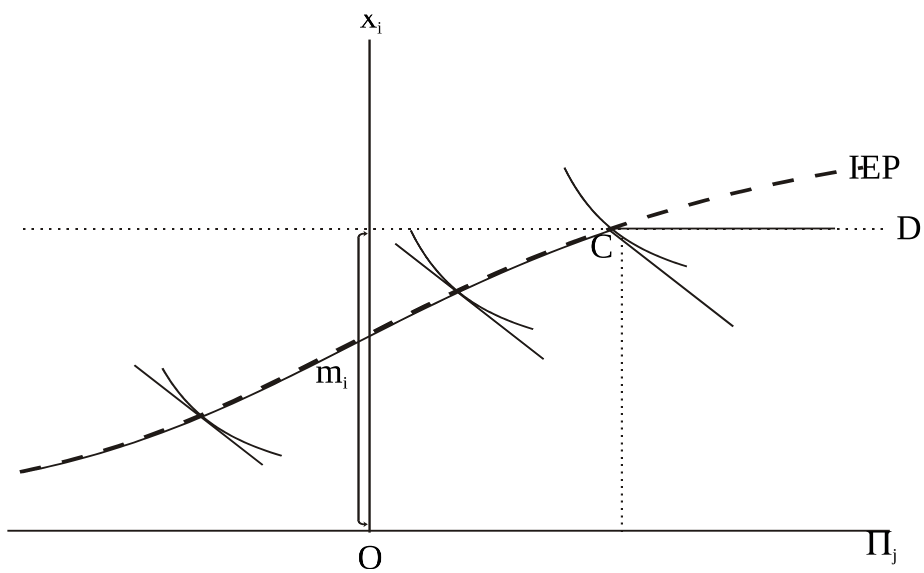


Figure 1: Preferences in (Π_j, x_i) space

$$\text{If } \Pi_{j\backslash i} \in \mathbf{A}_i,\ r_i\left(\Pi_j, m_i, c_i\right) = \widehat{r}_i$$

where $\widehat{r}_i$ is the unique solution to the equation

$$\Pi_j = \psi_i\left[m_i + c_i\left(\Pi_j - \widehat{r}_i\right)\right].$$

This last equation may be rearranged:

$$r_i\left(\Pi_j, m_i, c_i\right) = \frac{m_i - \psi_i^{-1}\left(\Pi_j\right)}{c_i} + \Pi_j.$$

It may be confirmed that the function $r_i\left(\Pi_j, m_i, c_i\right)$ is well-defined and continuous in Π_j and is monotonic decreasing if $\Pi_{j\backslash i} \in \mathbf{A}_i$.

Definition 1 *The function*

$$r_i\left(\Pi_j, m_i, c_i\right) = \max\left\{\frac{m_i - \psi_i^{-1}\left(\Pi_j\right)}{c_i} + \Pi_j, 0\right\}$$

is the replacement function of player i in group j.

Remark 1 *Consider any* Π_j. *Then there is a unique quantity* $Z \in [0, \Pi_j]$ *such that, if the amount* Z *were subtracted from the quantity* Π_j, *player* i*'s best response to the remaining quantity would exactly replace the quantity removed, and* $Z = r_i(\Pi_j)$. *Hence the label replacement function for* $r_i(\Pi_j, m_i, c_i)$.

To analyze this model, it is helpful to express the quantity Π_j in a slightly different way. Equation (1) may be written as

$$\begin{aligned} \Pi_j &= G_j + \theta \sum_{k \in J \setminus \{j\}} G_k \\ &= (1-\theta) G_j + \theta \sum_{k \in J} G_k \\ &= (1-\theta) G_j + \theta \Omega \end{aligned}$$

where $\Omega \equiv \sum_{k \in J} G_k$. Note that the 'global public good' Ω is defined by summing over every group's public good, including that of group j. Player i's 'replacement function' may therefore be written as

$$\begin{aligned} \widehat{g}_i &= r_i(G_j, \Omega : m_i, c_i, \theta) && (3) \\ &= \max \left\{ \frac{m_i - \psi_i^{-1}[(1-\theta) G_j + \theta \Omega]}{c_i} + [(1-\theta) G_j + \theta \Omega], 0 \right\}. && (4) \end{aligned}$$

In this expression, the intersecting interests of the various groups are entirely captured by the single global public good Ω. By contrast, the local public good G_j is of interest only to members of group j. This way of writing the player's replacement function permits a strategy, which we adopt in the next section, of analyzing equilibrium in a recursive, or hierarchical, manner.

The replacement function has several properties that follow directly from (3) and subsequently prove useful.

For the moment, hold Ω constant. Denote by $\underline{g}_i$ the contribution that player i $[i \in I_j]$, would choose to make when all other members of group j make no contribution:

$$r_i\left(\underline{g}_i, \Omega : m_i, c_i, \theta\right) = \underline{g}_i.$$

We will call $\underline{g}_i$ player i's participation value. The following proposition summarizes the most significant properties of player i's replacement function, $\gamma_i(G_j, \Omega : m_i, c_i, \theta)$:

Proposition 2 *The replacement function of player i in group j,* $\gamma_i(G_j, \Omega : m_i, c_i, \theta)$*, has the following properties:*

1. *For any given value of* Ω*, there exists a finite value* $\underline{g}_i$ *, player i's participation value, at which*

$$r_i\left(\underline{g}_i,\ \Omega : m_i, c_i, \theta\right) = \underline{g}_i.$$

2. $r_i(G_j, \Omega : m_i, c_i, \theta)$ *is defined for all* $G_j \geqq \underline{g}_i$*.*

3. $r_i(G_j, \Omega : m_i, c_i, \theta)$ *is continuous in* G_j *and* Ω*.*

4. $r_i(G_j, \Omega : m_i, c_i, \theta)$ *is everywhere nonincreasing in* G_j*, and is strictly decreasing wherever it is strictly positive.*

5. *If* $\theta > 0$*,* $r_i(G_j, \Omega : m_i, c_i, \theta)$ *is everywhere nonincreasing in* Ω*, and is strictly decreasing wherever it is strictly positive.*

6. *If* $\theta < 0$*,* $r_i(G_j, \Omega : m_i, c_i, \theta)$ *is everywhere nondecreasing in* Ω*, and is strictly increasing wherever it is strictly positive.*

Proof. Proofs of properties 1-4, where not already sketched above, follow the lines adopted in Cornes and Hartley [4]. Properties 5 and 6 follow readily by noting that $r_i(G_j, \Omega : m_i, c_i, \theta)$ depends on G_j and Ω entirely through its dependence on Π_j, and that $\Pi_j = (1-\theta) G_j + \theta\Omega$. ■

4 Modeling equilibrium in pure strategies

A Nash equilibrium is an allocation at which, in a well-defined sense, all players' choices are mutually consistent. The structure of the present model suggests that we distinguish between two types of consistency requirements, and that we analyze these requirements in two stages.

Stage 1: Group consistency Choose an arbitrary nonnegative value for the quantity Ω. which is held fixed throughout this stage and which we denote by $\overline{\Omega}$. A necessary condition for Nash equilibrium is that the total quantity of the public good associated with each group - say G_j -

be such that the sum of replacement values of group-j members equals that quantity. In short, we require that

$$\sum_{i \in I_j} r_i \left(\widehat{G}_j : \overline{\Omega}, m_i, c_i, \theta \right) = \widehat{G}_j \quad \text{for all } j \in J \tag{5}$$

There are N such conditions, one for each group, and each may be analyzed in isolation from the others. For a given value of $\overline{\Omega}$, we call a value $\widehat{G}_j$ 'group-j consistent' if it satisfies condition (5).

Stage 2: Overall consistency Suppose that, for an arbitrarily chosen $\overline{\Omega}$, we have found a set of group consistent values, $\left(\widehat{G}_1, \widehat{G}_2, ..., \widehat{G}_N\right)$. In general, there is no reason to suppose that their sum equals the arbitrarily chosen value $\overline{\Omega}$. For an allocation to be a Nash equilibrium, this additional requirement must also be met. That is, we require that

$$\sum_{j \in J} \widehat{G}_j = \Omega$$

where the quantities $\left(\widehat{G}_1, \widehat{G}_2, ..., \widehat{G}_N, \Omega\right)$ satisfy the group consistency requirements identified at the first stage.

An allocation that satisfies the group consistency conditions for every group, and also the overall consistency condition, is a Nash equilibrium.

4.1 Group consistency

Our analysis of group consistency closely mirrors that by Cornes and Hartley [4] of the standard pure public good model. Recall that, given $\overline{\Omega}$, group-j consistency requires that $\sum_{i \in I_j} r_i \left(\widehat{G}_j : \overline{\Omega} \right) = \widehat{G}_j$. The existence and uniqueness of such a value depends on the properties of the individual $\gamma_i(.)$ functions and of their sum.

The properties of continuity and monotonicity, stated in the proposition above, are preserved by the operation of addition. Therefore, the sum of replacement functions of all members of group j is itself a continuous function and is everywhere nonincreasing. Now consider the value of this sum when $G_j = \max \left\{ \underline{g}_i : i \in I_j \right\}$. Clearly,

$$\sum_{i \in I_j} \gamma_i \left(\max \left\{ \underline{g}_i : i \in I_j \right\}, \Omega : m_i, c_i, \theta \right) \geqq \max \left\{ \underline{g}_i : i \in I_j \right\}.$$

If this holds with strict equality, then $G_j = \max\left\{\underline{g}_i : i \in I_j\right\}$ satisfies the group consistency requirement for group j. This is a situation in which the player who is choosing her participation value is the sole contributor within group j. If it is a strict inequality, continuity and monotonicity ensure that there exists a unique higher value of G_j that equals the sum of associated replacement values of players in group j.

Figure 2 shows an example of a 3-player group. It shows the graphs of

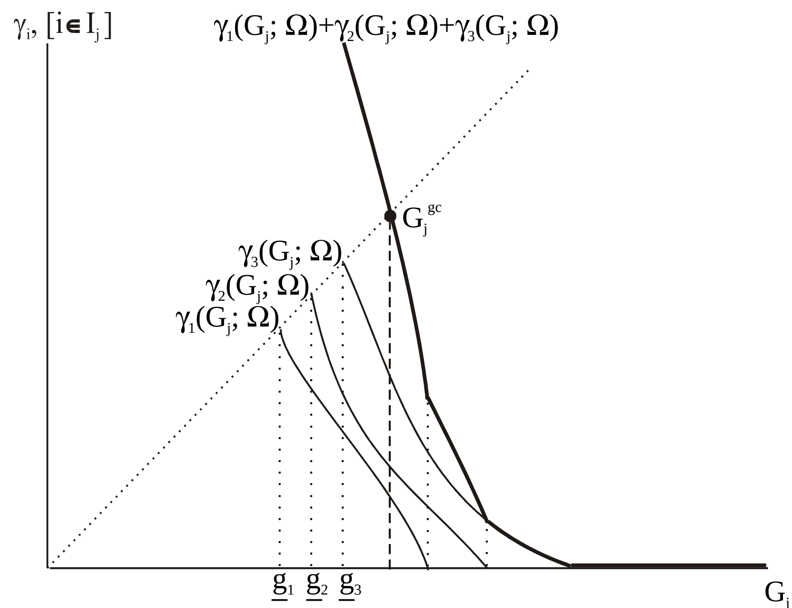


Figure 2: $\gamma_i\,(.)$ functions and group-j consistency

the players' replacement functions. Their vertical sum is indicated by the thicker line. The unique group-consistent value of G_j is the value G_j^{gc}, the value at which $\sum_{i\in I_j}\gamma_i\left(G_j^{gc},\Omega : m_i, c_i, \theta\right) = G_j^{gc}$. This analysis holds for all $j = 1, 2, ..., N$. We will write the function that describes this dependence of G_j^{gc} on Ω as $\Gamma_j\,(\Omega)$.

We can summarize the argument up to this point as follows:

Proposition 3 *For a given value of Ω, there exists at most a single set of group-consistent values $\mathbf{G}^{gc} \equiv (G_1^{gc}, G_2^{gc}, ..., G_N^{gc})$.*

4.2 Overall consistency

We now show that there exists a unique value of Ω for which the associated set of group-consistent values also satisfy the overall consistency requirement. We do so by considering the properties of the functions $\Gamma_j(\Omega)$, $j \in I$. We show that, for all $j \in J$, (i) if $\theta > 0$, $\Gamma_j(\Omega)$ is everywhere nonincreasing, and (ii) if $\theta < 0$, then $\Gamma_j(\Omega)$ is everywhere nondecreasing, for all j. In the latter situation, we also establish an upper bound on its slope.

Consider again the group-j consistency requirement:

$$\sum_{i \in I_j} \gamma_i (G_j, \Omega : m_i, c_i, \theta) = G_j$$

Clearly, the right hand side is strictly increasing in G_j. We have already shown that $\gamma_i(.)$ is nonincreasing in G_j. Therefore the left hand side of (5) is nonincreasing in G_j. Now consider an increase in Ω.

If $\theta > 0$,an increase in Ω implies an increase in Π_j, which implies a reduction in $\gamma_i(.)$ for all $i \in I_j$. Thus the left hand side of (5) falls. In Figure 2, the graphs of the individual, and therefore the group, replacement functions shift downwards. The group-j consistent value of G_j therefore falls. Thus the function $\Gamma_j(\Omega)$ is increasing.

If $\theta < 0$,an increase in Ω implies a reduction in Π_j, which in turn implies an increase in $\gamma_i(.)$ for every group member who is a positive contributor. Thus the left hand side of (5) increases. In Figure 2, the graphs of the individual and group replacement functions shift upwards. The group-j consistent value of G_j must therefore rise. Thus the function $\Gamma_j(\Omega)$ is decreasing.

In the event of $\theta < 0$, so that $\Gamma_j(\Omega)$ is increasing, the assumption that $\theta > -\frac{1}{N-1}$ places an upper bound on its numerical value. Consider a small change in Ω. Differentiating (5),

$$\sum_{i \in I_j} \left[\frac{\partial \gamma_i(.)}{\partial G_j} dG_j + \frac{\partial \gamma_i}{\partial \Omega} d\Omega \right] = dG_j$$

Now recall from (3) that G_j and Ω affect the value of $\gamma_i(.)$ solely by virtue of being constituent parts of Π_j. Since $\Pi_j = (1-\theta) G_j + \theta\Omega$, it follows that $\frac{\partial \gamma_i}{\partial \Omega} = \frac{\theta}{1-\theta} \frac{\partial \gamma_i(.)}{\partial G_j}$. Substituting into the above expression,

$$\sum_{i \in I_j} \left[\frac{\partial \gamma_i(.)}{\partial G_j} dG_j + \frac{\theta}{1-\theta} \frac{\partial \gamma_i(.)}{\partial G_j} d\Omega \right] = dG_j$$

Rearranging,

$$\begin{aligned}\frac{dG_j}{d\Omega} &= \Gamma_j'(\Omega) = \left(\frac{\theta}{1-\theta}\right)\left(\frac{\sum_{i\in I_j}\frac{\partial\gamma_i(.)}{\partial G_j}}{1-\sum_{i\in I_j}\frac{\partial\gamma_i(.)}{\partial G_j}}\right) \\ &= -k\frac{\theta}{1-\theta}.\end{aligned}$$

where $0 < k < 1$.Thus the assumption that $-\frac{1}{N-1} < \theta < 0$ implies that $0 < \frac{dG_j}{d\Omega} < \frac{1}{N}$.

We need to worry about the domain on which $\Gamma_j(\Omega)$ is defined. An economically meaningful situation requires that $\Gamma_j(\Omega) \leq \Omega$. Consider the group-$j$ consistent value of G_j when all other groups contribute zero to their public goods. Since, by definition, $\Omega \equiv \sum_{j=1}^{N} G_j$, this is the situation in which $\Gamma_j(\Omega) = \Omega$. Analogous to the approach in Cornes and Hartley, we define this quantity as the standalone value of group j, and denote it by $\underline{G}_j$. Conceivably, this value is zero [if it is zero for every group, then the situation in which $\Omega \equiv \sum_{j=1}^{N} G_j = 0$ is a Nash equilibrium]. More generally, we may expect it to be positive, at least for one group.

We have established that there is a nonnegative value, $\underline{G}_j$, with the property that $\Gamma_j\left(\underline{G}_j\right) = \underline{G}_j$. Moreover, (**??**) implies that for arbitrary values Ω^1 and Ω^0 such that $\Omega^1 > \Omega^0$, $\Gamma_j(\Omega^1) - \Gamma_j(\Omega^0) \leqq \Omega^1 - \Omega^0$, with a strict inequality holding if $\Gamma_j(\Omega^1) > 0$. Thus the function $\Gamma_j(\Omega)$ is defined for all $\Omega \geq \underline{G}_j$.

The function $\Gamma_j(\Omega)$ may be thought of as a group level replacement function. It describes the unique value of G_j that is, in the sense described above, consistent with any given nonnegative value of Ω.

4.3 Nash Equilibrium - Existence and Uniqueness

We have established that a necessary condition for an allocation to be a Nash equilibrium is that there be group consistency for every group. We have also shown how the group consistent values of the G_j's are related to the aggregate, Ω. Finally, we have argued that an overall consistency condition is also required at equilibrium. We now draw on our analysis of the $\Gamma_j(\Omega)$ relationships to show that there exists a unique Nash equilibrium in pure strategies.

Recall that the overall consistency condition for a Nash equilibrium requires that $\sum_{j=1}^{N} \Gamma_j(\Omega) = \Omega$. The left hand side of this condition is defined for all $\Omega \geq \max\{\underline{G}_1, \underline{G}_2, ..., \underline{G}_N\}$. The left hand side is an 'aggregate group level replacement function', defined as the sum of individual group functions. Its domain is $[\max\{\underline{G}_1, \underline{G}_2, ..., \underline{G}_N\}, \infty]$. Moreover, we can place an upper bound on its slope. Each of the N $\Gamma_j(\Omega)$ functions is either downward-sloping or, if it is upward-sloping, has slope less than $\frac{\theta}{1+\theta}$. Hence, the aggregate function cannot have a positive slope greater than $\frac{\theta N}{1+\theta}$. Consequently, we can state the following proposition:

Proposition 4 *If* $-\frac{1}{N-1} \leq \theta \leq 1$, *there is at most a single value of* Ω *for which* $\sum_{j=1}^{N} \Gamma_j(\Omega) = \Omega$.

Proof. It is convenient to distinguish between three situations, according to the value of the parameter θ.

1. If $0 < \theta \leq 1$, then $\Gamma_j(\Omega)$ is monotonically decreasing $\forall j \in I$. Thus as Ω increases, $\sum_{j=1}^{N} \Gamma_j(\Omega)$ cannot increase, and will fall wherever it is strictly positive. Thus, there can be at most a single value of Ω at which $\sum_{j=1}^{N} \Gamma_j(\Omega) = \Omega$.

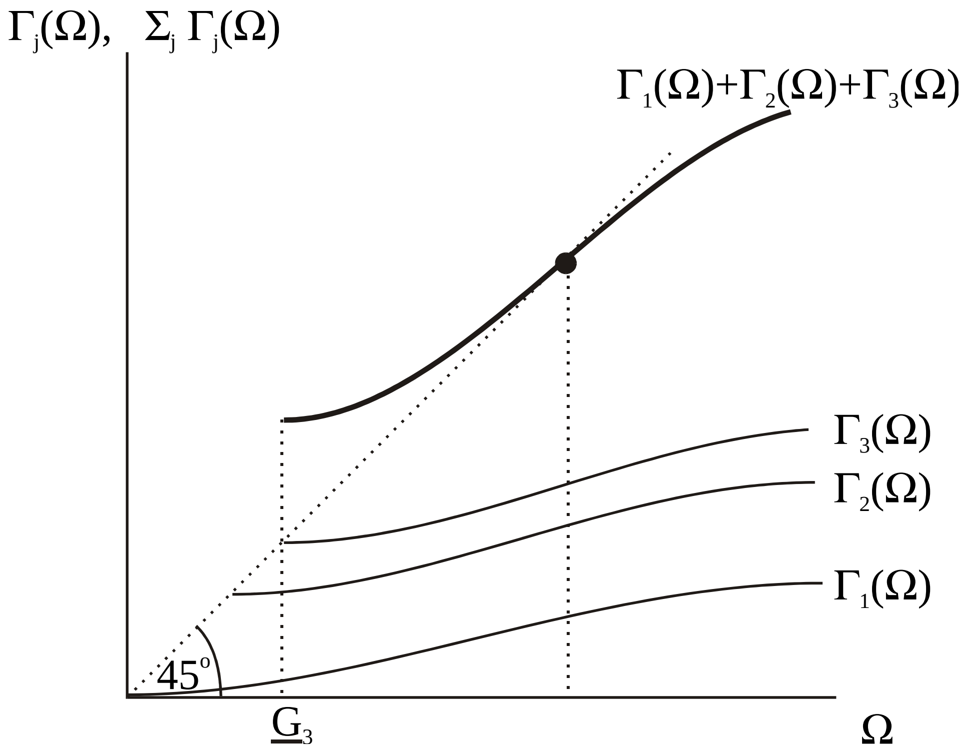
2. If $\theta = 0$, then there are N pure public good 'island' economies. It is already established that such an economy has at most a single value of Ω at which $\sum_{j=1}^{N} \Gamma_j(\Omega) = \Omega$.

3. If $-\frac{1}{N-1} < \theta < 0$, then then $\Gamma_j(\Omega)$ is monotonically increasing $\forall j \in I$. However, the upper bound to the slope of each $\Gamma_j(\Omega)$ implies that the function $\Psi(\Omega) \equiv \Omega - \sum_{j=1}^{N} \Gamma_j(\Omega)$ is everywhere monotonic increasing. Thus it has, at most, a unique solution.

■

Remark 2 *The precise value* $\theta = -\frac{1}{N-1}$ *provides a natural* N*-group extension of the two-group model, in which we assumed that the public good enjoyed by group 1 is* $\Pi_1 = G_1 - G_2$. *If there are* N *groups, and if* $G_2 = G_3 ... = G_N$, *the public good enjoyed by group 1 is* $\Pi_1 = G_1 - \theta(N-1)G_2$. *Clearly,* $\theta = \frac{1}{N-1} \Longrightarrow \Pi_1 = G_1 - G_2$.

Figure 3 shows the $\Gamma_j(\Omega)$ functions, and their summation, in a 3-group model in which $-\frac{1}{N-1} < \theta < 0$. Their sum is graphed by the thicker line. The aggregate group level replacement function is defined for all $\Omega \geq \underline{G}_3$.



Since its slope is everywhere less than 1, it follows that there can be at most a single point of intersection between its graph and the 45 ray through the origin. This is the unique equilibrium that we claim exists:

Proposition 5 *Under assumptions A1, A2, A3, and if* $-\frac{1}{N-1} \leq \theta \leq 1$*, then there exists a unique Nash equilibrium in pure strategies.*

5 Comparative Static Responses

5.1 Comparative statics of income changes

Consider player i in group j. Recall that i's replacement function is

$$r_i\left(G_j, \Omega : m_i, c_i, \theta\right) = \max\left\{\frac{m_i - \psi_i^{-1}\left[(1-\theta)\,G_j + \theta\Omega\right]}{c_i} + \left[(1-\theta)\,G_j + \theta\Omega\right], 0\right\} \tag{6}$$

Consider the implications of a change in i's money income. Denote her income before and after the change as m_i^0 and m_i^1 respectively. Let $\Delta m_i \equiv m_i^1 - m_i^0 > 0$ and denote the response of i's replacement value to her income change when other arguments remain constant by $\Delta_m r_i \equiv r_i(m_i^1 :) - r_i(m_i^0 :)$. Then inspection of 6 shows that

$$\Delta_m r_i \leqq \frac{\Delta m_i}{c_i}$$

with strict equality holding if $r_i(m_i^0 :) > 0$. All comparative static responses to income changes and redistributions flow from this simple inequality.

An immediate corollary is

Corollary 6 *Let players h and i both be members of the same group, j. Let both be positive contributors to G_j before and after a money income transfer from h to i: $\Delta m_h = -\Delta m_i > 0$. Then*

$$\Delta_m r_h + \Delta_m r_i \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0 \text{ according to whether } c_h \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} c_i.$$

This has immediate implications for equilibrium responses to income redistribution:

Proposition 7 *Let players h and i both be members of the same group, j. An income transfer from player h to i that leaves the set of positive contributors in group j unchanged raises the equilibrium level of G_j, leaves it unchanged, or reduces it, according to whether $c_h >, =$ or $< c_i$.*

The special case associated with the situation in which all players in group j have the same unit costs is the well-known neutrality proposition:

Proposition 8 ***[Neutrality]*** *If all members of group j have the same unit costs as public good contributors, then any income redistribution amongst the set of positive contributors in group j that leaves that set unchanged has no effect on the Nash equilibrium.*

Recent literature on global public goods takes seriously the possibility that unit costs differ across contributors. In this case, intragroup income redistribution has interesting normative implications:

Proposition 9 *Let players h and i both be positive contributors within the same group, j, and assume that $c_h > c_i$. Then income redistribution from h to i that leaves the set of contributors within group j unchanged*

1. *Increases the equilibrium value of G_j,*

2. *Increases the equilibrium utility of every member of group j.*

Clearly, if $c_h < c_i$, the word 'increases' should be changed to 'decreases' in the two parts of this proposition.

There is an interesting, though very special, situation in which a less familiar and more dramatic neutrality property holds. Assume that all players in the game have the same unit cost, which we may then equate to unity without loss of generality. Assume further that $\theta = -\frac{1}{N-1}$. This is the most extreme negative value for which we can be sure that the game is well-behaved. In the neighborhood of a given initial equilibrium, denote by M_j^+ the total income of those members in group j who are choosing to make strictly positive contributions to G_j. We denote the set of strictly positive contributors to G_j by I_j^+. Then the following proposition holds:

Proposition 10 ***[Neutrality*]*** *Let $\theta = -\frac{1}{N-1}$ and $c_i = c$ for all $i \in I^+$. Then an increase in incomes of positive contributors that gives each group of strictly positive contributors the same increase in aggregate income - that is, $\Delta M_j^+ = \Delta M$ for all $j \in J$ - has no effect on equilibrium quantities.*

To see this, note that

$$r_i(\Pi_j, m_i, c_i) = m_i - \psi_i^{-1}((1-\theta)G_j + \theta\Omega) + ((1-\theta)G_j + \theta\Omega) \quad \forall i \in I_j^+, j \in J$$

Suppose that, in response to the income change, each contributor increases her contribution by an amount

$$\Delta r_i = \Delta m_i$$

where

$$\sum_{i \in I_j} \Delta m_i = \Delta M_j = \Delta M \text{ for all } j \in I.$$

Then

$$\Delta G_j = \sum_{i \in I_j} \Delta r_i = \sum_{i \in I_j} \Delta m_i = \Delta M \text{ for all } j \in I$$

and

$$\Delta\Pi_j = ((1-\theta)\,\Delta M + \theta N \Delta M)$$

Recall that, by assumption, $\theta = -\frac{1}{N-1}$. Substitution yields

$$\Delta\Pi_j = \left[\left(1+\frac{1}{N-1}\right)\Delta M - \frac{1}{N-1} N\Delta M\right] = 0 \text{ for all } j \in I.$$

Thus, the public good consumed by each individual is unchanged. So too is that individual's private good consumption. Since relative prices are constant by assumption, the resulting allocation remains the unique equilibrium.

This is a notable property. It implies that under the present assumptions, starting from a Nash equilibrium, no finite amount of extra resources given to contributors can produce Pareto improvement. The extra resources are entirely spent on the guns, and not at all on the butter. Moreover, in such a world, a less equally distributed increase in total income may improve the welfare of members of some groups, but only at the cost of harming members of other groups.

We have already emphasized that this is an extreme case. We remind the reader that we have assumed that $\theta = -\frac{1}{N-1}$. If instead we let $-\frac{1}{N-1} < \theta < 0$, then at least some of the extra resources will go into private good consumption, and balanced income growth can achieve Pareto improvement in the new Nash equilibrium. However, even in this less extreme situation, unbalanced income growth will not generally benefit all. The various equilibrium implications of an increase in the income of a member of group j are set out in the following propositions:

Proposition 11 *An increase in m_i, $i \in I_j^+$, has the following equilibrium consequences for members of group j:*

- *player i's equilibrium contribution will rise,*
- *the equilibrium value of G_j will rise,*
- *the equilibrium utility of all members of group j will rise.*

The consequences of such an income increase are, of course, transmitted to members of other groups. The directions of these changes will depend on the assumed sign of the parameter θ. If intergroup externalities are beneficial, we have

Proposition 12 *Assume that* $0 < \theta \leq 1$*. An increase in* m_i*,* $i \in I_j^+$*, has the following equilibrium consequences for members of groups other than group* j*:*

- *equilibrium contributions of all existing positive contributors in groups other than group* j *will fall,*
- *therefore the equilibrium value of* G_k *,*$k \in J \setminus j$*, will not rise, and will generally fall,*
- *the equilibrium value of* Π_k*,*$k \in J \diagdown \{j\}$*,will rise, and the equilibrium utility of all members of group* k *,*$k \in J \diagdown \{j\}$*, will rise.*

However, if the intergroup externalities are harmful, we have

Proposition 13 *Assume that* $-\frac{1}{N-1} \leq \theta < 0$*. An increase in* m_i*,* $i \in I_j^+$*, has the following equilibrium consequences for members of groups other than group* j*:*

- *equilibrium contributions of all existing positive contributors in groups other than group* j *will rise,*
- *therefore the equilibrium value of* G_k *,*$k \in J \setminus j$*, will not fall, and will generally rise,*
- *the equilibrium value of* Π_k*,*$k \in J \diagdown \{j\}$*,will fall, and the equilibrium utility of all members of group* k *,*$k \in J \diagdown \{j\}$*, will fall.*

Regardless of the sign of θ, the equilibrium level of the global public good - or bad - Ω will rise.

6 A Numerical Example

6.1 The example

There are two groups [alliances A and B], each with 10 members.

Preferences are Cobb-Douglas: $u_i = x_i \Pi_j = x_i \left(G_j - \frac{G_k}{2} \right), i \in I_j$.

The groups have conflicting interests: $\theta = -\frac{1}{2}$.

Each member of alliance A has an income of 12 units.

Within alliance B, 2 members each have an income of 36 units. Each of the remaining 8 members has 5 units. [Thus the aggregate income of alliance B is less than that of alliance A].

6.2 Equilibrium in the example

The replacement function of player i in alliance j is

$$\widehat{g}_i = \max\left\{m_i - \frac{3}{2}G_j + \frac{1}{2}\Omega, 0\right\}.$$

Within alliance A all members are identical and make the same choice in equilibrium. Thus, at an equilibrium,

$$\widehat{g}_i = 12 - \frac{3}{2}\left(10\widehat{g}_i\right) + \frac{1}{2}\Omega, \quad i \in I_1$$

$$\Longrightarrow \widehat{g}_i = \frac{1}{32}\Omega + \frac{3}{4}, \quad i \in I_1$$

Turning to alliance B, inspection reveals that the dropout point of the low income members is less than that of the high income members. Thus, there are three posssible patterns of contribution. Either all members contribute, or only the two high income members contribute, or none contribute. Consider the replacement functions of the two high income members of alliance B. At allocations that generate positive contributions by these two, their replacement functions are

$$\widehat{g}_i = 36 - \frac{3}{2}G_2 + \frac{1}{2}\Omega, \quad i \in I_2.$$

Suppose that their 8 fellow members are contributing zero. Then $G_2 = 2\widehat{g}_i, i \in I_2$, and

$$\widehat{g}_i = \frac{1}{8}\Omega + 9, \quad i \in I_2.$$

If, indeed, the 8 low income members of alliance B are noncontributors, then the equilibrium is obtained by noting that, at equilibrium,

$$\begin{aligned}\Omega &= G_1 + G_2 \\ &= 10\left(\frac{1}{32}\Omega + \frac{3}{4}\right) + 2\left(\frac{1}{8}\Omega + 9\right)\end{aligned}$$

$$\Longrightarrow \Omega = 58.286.$$

It may be confirmed that, at such an equilibrium, the remaining 8 low income members of alliance B will prefer to make zero contributions. Thus we have found the true Nash equilibrium. Of particular interest in this example are the equilibrium payoff levels. They are:

Of all alliance A members:

$$u = (12 - 2.5714)\left(25.714 - \frac{32.572}{2}\right) = 88.893$$

Of the 2 high income alliance B members:

$$u = (36 - 16.286)\left(32.572 - \frac{25.714}{2}\right) = 388.66$$

Of the 2 low income alliance B members:

$$u = (5)\left(32.572 - \frac{25.714}{2}\right) = 98.575$$

Thus, even though alliance B has a lower level of aggregate income than alliance A, its unequal distribution amongst its members leads to an equilibrium in which every alliance B member is better off than every alliance A member.

7 An Extension to Increasing Costs

Our formal Neutrality* proposition is striking. It implies that, beyond a certain level, additional resources do not produce Pareto improvement. If each group receives the same increment of additional income, then at the new equilibrium all of this income is spent on generating additional public good for that group, at the expense of others. At the margin, the groups are essentially trapped in a zero-sum game. One might argue that this is too dramatic a conclusion. One obvious escape route notes that neutrality no longer holds if $\theta > -\frac{1}{N-1}$. Less severe negative externalities dampen, but do not completely negate, the effects of a general enhancement of resource endowments. This is empirically perfectly plausible, especially since $\theta = -\frac{1}{N-1}$ is, as we pointed out, a very specific case. However, there is another escape route that is, at least to a theorist, as interesting [because it is rather less obvious]. This is to acknowledge that, in the world of production, the Ricardian constant cost technology is a very special case. Maybe it is more reasonable to suppose that the marginal cost of generating the public good increases with an individual's contribution level. Let us consider the implications of this assumption. We now write player i's budget constraint as

$$x_i + C_i(g_i) = m_i$$

where $C_i(g_i)$ is the total cost of contributing g_i units to G_j. We assume

A3′ $C_i(0) = 0,\ C_i'(g_i) > 0,\ C_i''(g_i) > 0.$

First, let us briefly remind ourselves of the implications of having constant unit costs, but with $c_i \neq 1$. As above, a contributor's public good contribution is. $g_i = \frac{m_i - \psi_i^{-1}(\Pi_j)}{c_i} + \Pi_j$. For a given level of Π_j, an increase in i's income leads to an increase in her contribution of

$$dg_i = \frac{\partial r_i}{\partial m_i} dm_i = \left(\frac{1}{c_i}\right) dm_i. \qquad (7)$$

Now return to the situation with increasing costs. This implies that i's private good consumption is

$$x_i = \xi_i(m_i, \Pi_j), 0 < \frac{\partial \xi_i(m_i, \Pi_j)}{\partial m_i} < 1, \frac{\partial \xi_i(m_i, \Pi_j)}{\partial \Pi_j} > 0,$$

Thus

$$\begin{aligned} C_i(\widehat{g}_i) &= m_i - \xi_i(m_i, \Pi_j) \\ \widehat{g}_i &= \gamma_i(.) = f_i(m_i - \xi_i(m_i, \Pi_j)) \end{aligned}$$

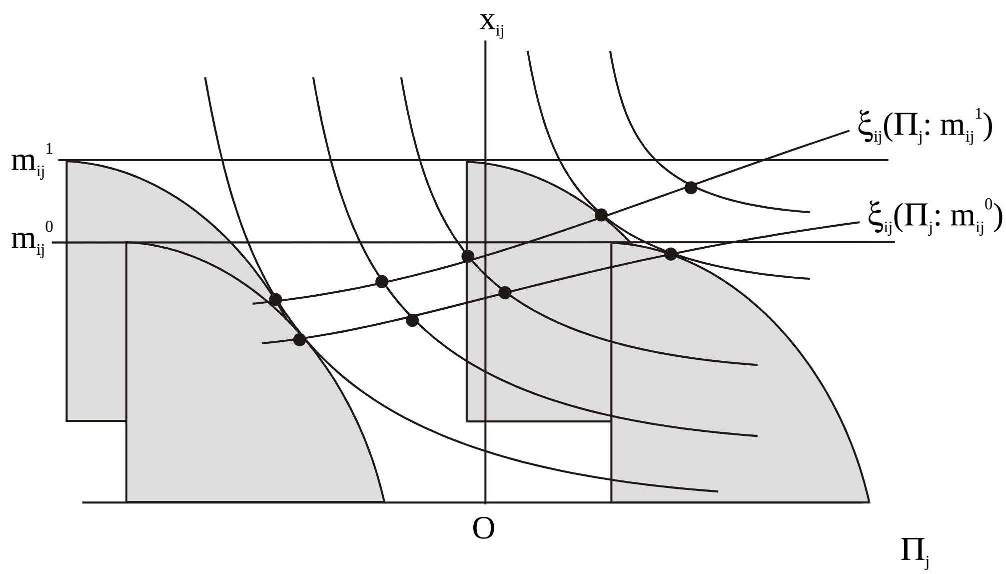
where $f_i(.) = C_i^{-1}(.)$, so that $f_i'(.) > 0$ and $f_i''(.) < 0$.

Now consider an infinitesimal change in m_i. The response of $\widehat{g}_i$ is

$$\frac{\partial \gamma_i(.)}{\partial m_i} = f_i'(.)\left[1 - \frac{\partial \xi_i(.)}{\partial m_i}\right] \qquad (8)$$

The term $f_i'(.)$ is the analogue of the term $\frac{1}{c_i}$ in (7). The term in square brackets is the new feature. Since $0 < 1 - \frac{\partial \xi_i(.)}{\partial m_i} < 1$, it indicates that only a fraction of the income increment goes to extra public good contribution. A part of it is devoted to additional private good consumption. Figure 4 summarizes the way in which the income increase shifts the income expansion path in space upward.

Thus, even if the new equilibrium implies an unchanged value of Π_j, player i enjoys an increase in her private good consumption, and in her equilibrium utility.



8 Conclusion

This paper has developed a formal model that can accommodate both the intragroup free rider problem that has interested students of collective action, and also the intergroup interdependence that is usually suppressed in such models. The model has deliberately been pared down as far as possible. However, it may be extended in several promising directions. Our simple assumption of additivity may be replaced by a more general additively separable structure. Furthermore, it would be interesting to consider cases in which the public good enjoyed by a group is not a pure nonexcludable good for its members, but can be distributed amongst them according to specified sharing rules. The precise nature of such rules will influence the incentives for members to contribute. Lastly, many groups consist, in turn, of subgroups, that in turn consist of..., and so on. Such extensions, though complicating matters, are in principle capable of being analyzed using the hierarchical approach adopted in this paper.

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