

Share Equilibrium in Local Public Good Economies*

by

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Abstract: We define a concept of share equilibrium for local public good (or club) economies where players may have preferences over the membership of their jurisdiction. A share equilibrium specifies one share index for each player in an economy. These indices determine each player's cost shares in any jurisdiction that he might join. We establish a relation between share equilibrium and the core and also prove an existence result and provide an axiomatic characterization of share equilibrium using consistency properties.

1 An introduction to share equilibrium

This paper introduces the concept of a 'share equilibrium' in the context of an economy with local public goods. A share equilibrium endogenously determines a share index for each player in the economy. The share index of a player determines what he pays for the public good in his allocated jurisdiction and also what he would pay in any other jurisdiction he might join.¹ In many situations involving clubs, local public goods, or group activities, the shares of the costs of such activities are determined by the relative share indices of the individuals involved in the activity. There are numerous examples. We are told that in Korea, when a group of professors go to a restaurant for dinner, the senior professor pays. In some condominium associations, the shares of the costs of maintenance of the condominium are determined by relative sizes of condominium units, which could be interpreted as share indices

¹One might compare equilibrium share indices to Walrasian equilibrium prices. An equilibrium price vector determines the cost of the equilibrium bundle purchased by each agent and also the cost of any alternative bundle that he might consider purchasing – but does not wish to since, given the constraints that he faces, he would be no better off. Similarly, equilibrium share indices determine the cost share an agent would have to pay in each jurisdiction that he might join.

of the owners of the units. If one subscribes to *Econometrica* or a number of other academic journals, the cost of the journal to an individual depends on the share index of the individual – rank within own university and the country in which the university is located. Such examples are sufficiently common to motivate the appeal of the share equilibrium, a new concept introduced in this paper.

The share equilibrium has a second motivation, which is to extend the ratio equilibrium concept to economies with local public goods. The ratio equilibrium is an appealing concept and well reflects the ideas of cost sharing in Lindahl (1919). As has been observed in the literature, in the presence of constant returns to scale in production of (pure) public goods, the Lindahl equilibrium, as described in his original paper, yields the same outcomes as the equilibrium concept defined in Samuelson (1954). The two equilibrium concepts however, may yield different outcomes in general, because they have different underlying philosophies and therefore are fundamentally different. This becomes clear when one considers the properties of the equilibrium concepts. In a previous paper we have shown that when the ideas of Lindahl are translated into axioms, we find a set of axioms that are satisfied by the ratio equilibrium but not by the Samuelson equilibrium. The purpose of the current paper is to extend the ratio equilibrium, which is only defined for (pure) public good economies, to local public good economies. We justify the share equilibrium as the 'right' extension of the ratio equilibrium by providing an axiomatic characterization of the share equilibrium using axioms analogous to those in the axiomatic characterization of the ratio equilibrium.

In the remainder of this section of the paper, we provide a more detailed description of the equilibrium concept, the model, and the results. Since a share equilibrium is a new concept, there are many open questions; in a concluding section, we report on research in progress dealing with some of

these.

Recall that a (pure) public good is a commodity that can be consumed in its entirety by all players in an economy; consumption of the good by an additional player does not decrease the amount available to the other members of the society. Thus, unlike the situation for private goods, cost-sharing rules for public goods cannot be determined by competition between players for the available supplies of the commodity. In an economy with local public goods, as suggested by Tiebout (1956), public goods are subject to possible congestion and also to exclusion. We define a jurisdiction as a subset of players who provide the public goods exclusively for the use of the members of that subset. If there are sufficiently many players local public goods can be (near) optimally provided to jurisdictions that are “small” relative to the total population. We note that in much of the literature there is no distinction between club goods and local public goods and between jurisdictions and clubs.

Various solutions to the problem of allocation of costs of public good provision have been proposed. The most well-known is perhaps the Lindahl equilibrium, introduced in Lindahl (1919) and formalized in Samuelson (1954), Johansen (1963), and Foley (1970).² As formalized by Samuelson, the Lindahl equilibrium permits individuals to pay personalized prices for public goods. In contrast, however, according to our reading of Lindahl’s paper, players pay shares of the total costs and, in equilibrium, these shares must satisfy the property that the amount of public good provided is the same for all individuals, given their cost shares.³ Other papers that have

²See also Mas-Colell (1980) where the concept of public goods was extended to public projects.

³In a recent paper, van den Nouweland, Tijs, and Wooders (2002) axiomatize the ratio equilibrium cost-sharing rule by means of consistency properties, where the consistency property used is very much in the spirit of Lindahl’s original work.

taken approaches in a similar spirit, with individuals paying shares of costs rather than per unit prices, include Kaneko (1977a,b) and Mas-Colell and Silvestre (1989,1991).

In the current paper, we study local public good economies (or, in other words, economies with clubs). The defining features of a local public good economy are that the public good can be consumed in its entirety by all members of a jurisdiction while non-members are excluded from consumption. In a share equilibrium for an economy with local public goods, the share index of a player determines his share of costs in any jurisdiction he might join. Roughly, a share equilibrium includes a specification of a partition of the set of players into jurisdictions, a production of local public goods for each jurisdiction, an allocation of private goods for each player, and a share index for each player. The relative share of a player in any jurisdiction to which he might belong is given by his share index divided by the sum of the share indices of all the players in the jurisdiction. The relative share of a player in a jurisdiction gives his share of the costs of public good provision in that jurisdiction.

We stress that the share equilibrium endogenously determines a share index for each player in the economy. The share index of a player determines not only what he pays for the public good in his allocated jurisdiction but also what he would pay in any other jurisdiction he might join. Players with a higher share index pay a larger share of the cost of local public good production than players with a lower share index in the same jurisdiction, as their cost shares are proportional to their share indices. Hence, a higher share index corresponds to a larger cost share. This is not unlike the way in which public schools (a local public good) are financed through property taxes; property taxes, in turn, are proportional to the value of one's home, and the value of one's home is related to one's income. Many people choose

to live in specific neighborhoods based, at least in part, on the quality of the public school (i.e., the level of local public good provision) and the property taxes in the neighborhood. These, in turn, are influenced by the composition of the neighborhood in terms of income levels and property values.

A point we wish to emphasize is that share indices reflect relative shares of costs that players in various jurisdictions shoulder and thus a share equilibrium depends on only one set of indices — we associate with each decision-making player i a *single* index s_i that reflects the player’s relative burden in all possible jurisdictions and for all possible levels of public good production. We note that, in contrast, extensions of the Lindahl equilibrium (as formalized in Samuelson 1954 and Johansen 1963) to local public good economies depend on the use of a per-unit price for each player for the local public good for each jurisdiction he might possibly join (cf. Wooders 1997) while extensions of the Walrasian equilibrium to local public goods economies depend on a price for each player for each jurisdiction that he may join and for each level of public good that can be produced in that jurisdiction (cf. Allouch and Wooders 2008). Cost-sharing notions of equilibrium to economies with local public goods as in, for example, Konishi, Le Breton and Weber (1997, 1998), treat equal cost sharing of all members of each jurisdiction for the local public good provided in that jurisdiction.⁴

In this first paper, we define the share equilibrium and study its properties and its relation to the core. We show that a share equilibrium leads to jurisdictions and consumption levels of local public good and private good that are stable against group formation and we prove the existence of share equilibrium for a set of symmetric local public good economies. We also pro-

⁴These authors and many others, including the recent contribution of Gravel and Thoron (2007), also treat taxation schemes where the tax imposed on an individual is a percentage of his wealth.

vide an axiomatic characterization of the share equilibrium.⁵ Our axiomatic characterization revolves around a consistency property that is an extension of the consistency property used to axiomatize equilibrium in pure public good economies in van den Nouweland, Tijs, and Wooders (2002). Hence, the axiomatization of the share equilibrium for local public good economies provides both a justification for this concept as the proper extension of the Lindahl equilibrium for pure public good economies and an evaluation of this concept on the basis of its properties.

2 Local public good economies

This section is devoted to formal definitions. We limit ourselves to economies with one public good and one private good. The results can easily be extended to economies with an arbitrary finite number of public goods. Doing so does not require any structurally different argumentation, but it would require more complicated notation. We prefer to avoid distracting technical matters and thus limit ourselves to economies with one public good and one private good.

A new element in our definition of local public good economies is the introduction of a set of decision-making players. The distinction between decision-making players and non-decision making players is made because, in defining reduced economies in Subsection 6.1, we must distinguish between subsets of players whose decisions are taken as given and other players who remain active in the decision-making process. To illustrate, consider the set of condominium owners in a building. When there is a meeting of the

⁵Axiomatic characterization of solution concepts is a well established approach. Recent contributions to the literature using such approaches include, for example, Moulin (2000), Dhillon and Mertens (1999), and Maskin (1999).

condominium home-owners association, some subset of owners may choose not to attend, knowing that they will still have to pay their shares of the costs of the association. These owners become non-decision makers, taking their shares of the condominium fees as given and relinquishing the power to decide on all matters effecting the association to the attendees.

Formally, a *local public good economy* is a list

$$E = \langle N, D; (w_i)_{i \in N}; (u_i)_{i \in N}; f_D \rangle,$$

where N (sometimes denoted $N(E)$) is the non-empty finite set of players in the economy, $D \subseteq N$ (or $D(E)$) is the set of decision-making players in the economy, $w_i \in \mathbb{R}_+$ is the non-negative endowment of player $i \in N$ of a private good, $u_i : \mathbb{R}_+ \times \mathbb{R}_+ \times 2^N \rightarrow \mathbb{R}$ is the utility function of player i , and $f_D : \mathbb{R}_+ \times 2^N \rightarrow \mathbb{R}_+$ is the cost function for the production of local public good in jurisdictions. The family of all public good economies is denoted by \mathcal{E} .

In an economy E , if player i is a member of jurisdiction $J \subseteq N$ ($i \in J$) and consumes an amount x_i of the private good and an amount y of the local public good provided in jurisdiction J , then player i enjoys utility $u_i(x_i, y, J)$. We assume that u_i is strictly increasing in both private and (local) public good consumption.

In a jurisdiction J , the cost of producing y units of the local public good borne by the decision-making players in J is $f_D(y, J)$ units of the private good. The cost function f_D is non-decreasing in the level of (local) public good with $f_D(0, J) = 0$ for each J . Hence, if each decision-maker $i \in J \cap D$ in jurisdiction $J \subseteq N$ contributes an amount z_i of the private good toward the production of the local public good, then, if the feasibility condition $f_D(y, J) \leq \sum_{i \in J \cap D} z_i$ is satisfied, bundle y of the local public good can be provided.

To illustrate the cost function and its interpretation, let N be the set of condominium owners in a building and let D be the set of owners who attend a particular meeting of the condominium home-owners association. In our terminology, the owners in $N \setminus D$ become non-decision makers. However, not attending the meeting does not absolve the absent owners from paying their shares of the costs of the association. The owners present at the meeting may approve new projects involving new costs to the association. For any new project, the costs taken into account by the owners in D are those faced by these owners, which is the fraction of the costs that is not covered by the absent owners. It is easiest to think about the cost $f_D(y, J)$ as the *residual cost* to the decision-making players in J of producing y , after subtracting the cost-shares of players in $J \setminus D$.⁶

A specification of a jurisdiction structure of the set of players, the levels of local public good provided in each jurisdiction, and private good consumptions is called a configuration. Hence, a *configuration* in a local public good economy E with set of players N is a vector

$$(\mathbf{x}, \mathbf{y}, \mathbf{P}) = ((x_i)_{i \in N}, (y_P)_{P \in \mathbf{P}}, \mathbf{P}),$$

where, for each $i \in N$, $x_i \in \mathbb{R}_+$ is the consumption of the private good by player i , \mathbf{P} is a partition of N into jurisdictions, and $y_P \in \mathbb{R}_+$ is the level of local public good provided in jurisdiction $P \subseteq N$ for each $P \in \mathbf{P}$. We denote the set of configurations in a local public good economy E with set of players N by $C(N)$. A configuration $(\mathbf{x}, \mathbf{y}, \mathbf{P}) \in C(N)$ is *feasible* if $f_D(y_P, P) \leq \sum_{i \in P \cap D} (w_i - x_i)$ for each $P \in \mathbf{P}$.

⁶We refer the reader to Subsection 6.1 for a more detailed motivation and explanation for the introduction of a set of decision-making players.

3 Share equilibrium

A share equilibrium for a local public good economy consists of a vector of share indices - one for each decision-making player in the economy - and a configuration. Share indices determine the share of each decision-making player of the cost of the production of local public good in all possible jurisdictions; if he is a member of jurisdiction $J \subseteq N$, a decision-making player $i \in D$ with share index s_i pays the share $s_i / \left(\sum_{j \in J \cap D} s_j \right)$ of the cost of local public good production in jurisdiction J . Hence, share indices determine the *relative* cost shares paid by the decision-making players in each jurisdiction that might possibly be formed. A set of share indices and a configuration constitute a share equilibrium if (1) every decision-making player's membership of a jurisdiction and consumption as specified by the configuration are utility-maximizing in his budget set as determined by his (relative) share and, moreover, (2) in every jurisdiction that is formed, all decision-making members demand the same level of local public good. This implies that, in equilibrium, given the share of the cost of local public good production that he has to shoulder in various jurisdictions as determined by the share indices, each decision-making player prefers the jurisdiction to which he is assigned and the level of local public good that is provided in that jurisdiction. Moreover, in a share equilibrium, the players agree to share the cost of local public good production according to their share indices. Hence, a share equilibrium is an equilibrium in three dimensions: the cost shares arising from the players' share indices, the jurisdictions formed, and a level of local public good production for each jurisdiction that is formed. Agreement on the share indices determining cost shares, formation of jurisdictions, and levels of local public good are inextricably linked.

Formally, for a local public good economy $E = \langle N, D; (w_i)_{i \in N}; (u_i)_{i \in N}; f_D \rangle$,

a set of share indices is a positive vector $\mathbf{s} = (s_i)_{i \in D} \in \mathbb{R}_{++}^D$.⁷⁸ For each player $i \in D$ and each jurisdiction $J \subseteq N$, player i 's relative share in J is $s_i^{J,D} := s_i / \left(\sum_{j \in J \cap D} s_j \right)$. Also, if \mathbf{P} is a partition of N , then for each $i \in N$ we denote the jurisdiction containing player i by $P(i)$, thus $i \in P(i) \in \mathbf{P}$.

A pair consisting of a set of share indices and a configuration $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ is a *share equilibrium* in economy E if for each $i \in D(E)$,⁹

1. $s_i^{P(i),D} f_D(y_{P(i)}, P(i)) + x_i = w_i$, and,
2. for all $(\bar{x}_i, \bar{y}, \bar{J}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times 2^N$ satisfying $i \in \bar{J}$ and $s_i^{\bar{J},D} f_D(\bar{y}, \bar{J}) + \bar{x}_i \leq w_i$, it holds that $u_i(x_i, y_{P(i)}, P(i)) \geq u_i(\bar{x}_i, \bar{y}, \bar{J})$.

The set of share equilibria of an economy $E = \langle N, D; (w_i)_{i \in N}; (u_i)_{i \in N}; f_D \rangle$ is denoted $SE(E)$.

Note that the share indices appear only in a relative manner, so that if $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ is a share equilibrium in an economy E , and $\alpha > 0$, then $(\alpha \mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ is also a share equilibrium in economy E .

⁷Assuming share indices to be positive is without loss of generality. It follows from condition 2 of share equilibrium and agents' utility functions being strictly increasing in local public good consumption that there can be no share equilibrium in which an agent has a share index equal to zero as this would imply that an agent always wants to consume more of the local public good, which is free to him. For similar reasons, it is impossible to have some agents be subsidized, i.e. have a negative share index.

⁸We could normalize the share indices to sum up to 1. However, this is unnecessary for any of our results and would complicate notation in the paper. Moreover, such a normalization is inherently arbitrary - for example, we may as well normalize the share indices such that they sum up to the number of players. Therefore, we have opted not to normalize the share indices at all, but let them be determined by the equilibrium conditions solely.

⁹The equality in 1 could be " \leq ", but using " $=$ " is without loss of generality as an agent would always be able to consume more of the private good and therefore increase his utility if he were not spending all of his endowment. Therefore condition 2 cannot be satisfied if $s_i^{P(i),D} f_D(y_{P(i)}, P(i)) + x_i < w_i$.

4 Share equilibrium and the core

In this section we explore relations between share equilibria and the core of a local public good economy.

The core of an economy is the set of configurations that are stable against deviations by groups of decision-making players. When a group of decision-making players deviates, they can form a new jurisdiction and within this jurisdiction the group can decide on a level of local public good to be provided and on a way to share the cost of its provision among the group members. Formally, the *core* of an economy $E = \langle N, D; (w_i)_{i \in N}; (u_i)_{i \in N}; f_D \rangle$ is the set of all configurations $(\mathbf{x}, \mathbf{y}, \mathbf{P}) \in C(N)$ such that for every $T \subseteq N$, private-good consumption levels $(\bar{x}_i)_{i \in T}$, and local public good level \bar{y} satisfying $f_D(\bar{y}, T) \leq \sum_{i \in T \cap D} (w_i - \bar{x}_i)$, it holds that if there exists a player $i \in T \cap D$ who is strictly better off, i.e. $u_i(x_i, y_{P(i)}, P(i)) < u_i(\bar{x}_i, \bar{y}, T)$, then there exists a player $j \in T \cap D$ who is strictly worse off, i.e. $u_j(x_j, y_{P(j)}, P(j)) > u_j(\bar{x}_j, \bar{y}, T)$.

Theorem 1 shows that every share equilibrium leads to a configuration in the core. That is to say, if a configuration is such that there exists a set of share indices that, together with that configuration, forms a share equilibrium, then the configuration is in the core of the economy.

Theorem 1. Let $E = \langle N, D; (w_i)_{i \in N}; (u_i)_{i \in N}; f_D \rangle \in \mathcal{E}$ be a local public good economy, and let the pair of share indices $(s_i)_{i \in D}$ and configuration $((x_i)_{i \in N}, (y_p)_{p \in \mathbf{P}}, \mathbf{P})$ be a share equilibrium in economy E . Then the configuration $(\mathbf{x}, \mathbf{y}, \mathbf{P})$ is in the core of E .

Proof. Take a coalition $T \subseteq N$ and an allocation of private and local public good consumption $((\bar{x}_i)_{i \in T}, \bar{y})$ satisfying

$$f_D(\bar{y}, T) \leq \sum_{i \in T \cap D} (w_i - \bar{x}_i).$$

We distinguish between $f_D(\bar{y}, T) = 0$ and $f_D(\bar{y}, T) > 0$.

Suppose $f_D(\bar{y}, T) = 0$ and suppose that for player $i \in T \cap D$ it holds that $u_i(x_i, y_{P(i)}, P(i)) < u_i(\bar{x}_i, \bar{y}, T)$. Then because $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in SE(E)$, we know by condition 2 of share equilibrium that $\bar{x}_i = s_i^{T,D} f_D(\bar{y}, T) + \bar{x}_i > w_i$. Together with $0 = f_D(\bar{y}, T) \leq \sum_{j \in T \cap D} (w_j - \bar{x}_j)$, this implies that there exists a player $k \in T \cap D$ for whom $\bar{x}_k < w_k$. However, player k can afford to be in jurisdiction T and consume w_k of the private good and \bar{y} of the local public good (because $s_k^{T,D} f_D(\bar{y}, T) + w_k = w_k$). Because $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ is a share equilibrium and the player's utility function is strictly increasing in private good consumption, we can derive

$$u_k(x_k, y_{P(k)}, P(k)) \geq u_k(w_k, \bar{y}, T) > u_k(\bar{x}_k, \bar{y}, T).$$

Hence, k would be strictly worse off by belonging to T with the proposed allocation. Conclude that $(\mathbf{s}, \mathbf{y}, \mathbf{P})$ is stable against deviations $((\bar{x}_i)_{i \in T}, \bar{y}, T)$ with $f_D(\bar{y}, T) = 0$.

Now suppose $f_D(\bar{y}, T) > 0$. Then $\sum_{i \in T \cap D} (w_i - \bar{x}_i) > 0$, and we can induce relative share indices for T given by

$$\tau_i := \frac{w_i - \bar{x}_i}{\sum_{j \in T \cap D} (w_j - \bar{x}_j)}$$

for all $i \in T \cap D$. Note that $\sum_{i \in T \cap D} \tau_i = 1$ and $\tau_i f_D(\bar{y}, T) + \bar{x}_i \leq w_i$ for all $i \in T \cap D$.

We compare the induced relative share indices for the decision-making members of T , $(\tau_i)_{i \in T \cap D}$, to their relative indices according to \mathbf{s} , which are given by

$$s_i^{T,D} := \frac{s_i}{\sum_{j \in T \cap D} s_j}$$

for each $i \in T \cap D$. We distinguish two cases.

Case 1. Suppose that $s_i^{T,D} = \tau_i$ for all $i \in T \cap D$. Then, for all $i \in T \cap D$,

$$s_i^{T,D} f_D(\bar{y}, T) + \bar{x}_i = \tau_i f_D(\bar{y}, T) + \bar{x}_i \leq w_i.$$

By condition 2 of the share equilibrium, we then know that $u_i(x_i, y_{P(i)}, P(i)) \geq u_i(\bar{x}_i, \bar{y}, T)$ for all $i \in T \cap D$. Hence, no decision-making player in T is strictly better off in jurisdiction T with consumption bundle (\bar{x}_i, \bar{y}) than he is in configuration $(\mathbf{x}, \mathbf{y}, \mathbf{P})$.

Case 2. Suppose that not $s_i^T = \tau_i$ for all $i \in T \cap D$. Since $\sum_{j \in T \cap D} s_j^{T,D} = \sum_{j \in T \cap D} \tau_j = 1$, it holds that $s_k^{T,D} < \tau_k$ for some $k \in T \cap D$. Such a player k would loose from belonging to T with the proposed allocation. To see this, first observe that

$$w_k \geq \tau_k f_D(\bar{y}, T) + \bar{x}_k > s_k^{T,D} f_D(\bar{y}, T) + \bar{x}_k.$$

Consider

$$\bar{z}_k := w_k - s_k^{T,D} f_D(\bar{y}, T) > \bar{x}_k.$$

Then

$$s_k^{T,D} f_D(\bar{y}, T) + \bar{z}_k = w_k.$$

It follows from condition 2 of the share equilibrium $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ and the assumption that a player's utility function is strictly increasing in private good consumption that

$$u_k(x_k, y_{P(k)}, P(k)) \geq u_k(\bar{z}_k, \bar{y}, T) > u_k(\bar{x}_k, \bar{y}, T).$$

Hence, k would be strictly worse off by belonging to T with the proposed allocation.

Based on Case 1 and Case 2, conclude that $(\mathbf{x}, \mathbf{y}, \mathbf{P})$ is also stable against deviations $((\bar{x}_i)_{i \in T}, \bar{y}, T)$ with $f_D(\bar{y}, T) > 0$. ■

5 Existence of a share equilibrium

We have defined share equilibrium because we are interested in extending the ratio equilibrium concept to local public good economies. Now that we have identified a new equilibrium concept, there are, of course, many questions that arise. One of these questions concerns existence. While we defer an extensive study of the existence of the share equilibrium to later work, we identify a set of sufficient conditions for existence of the share equilibrium in this section.

Players in a local public good economy have preferences over private and local public good consumption and over the jurisdiction members with which they share the local public good. A local public good economy is symmetric if all its players play the same role and have the same preferences and initial endowments. Note that, because players enter into one another's utility functions, this necessarily implies that players care only about how many players are in their jurisdiction and not about the identities of those members. It also implies that costs for producing the local public good in a jurisdiction only depend on how many players are in the jurisdiction. Technically, a local public good economy $E = \langle N, D; (w_i)_{i \in N}; (u_i)_{i \in N}; f_D \rangle \in \mathcal{E}$ is *symmetric* if all players participate as decision makers ($D = N$), there exists a $w \in \mathbb{R}_+$ such that $w_i = w$ for all $i \in N$, there exists a function $f : \mathbb{R}_+ \times \mathbb{N} \rightarrow \mathbb{R}_+$ such that $f_D(y, J) = f(y, |J|)$ for all $y \in \mathbb{R}_+$ and $J \subseteq N$, and there exists a function $u : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N} \rightarrow \mathbb{R}$ such that for each player $i \in N$, each jurisdiction $J \subseteq N$ with $i \in J$, each amount of private good x_i and amount of local public good y , it holds that $u_i(x_i, y, J) = u(x_i, y, |J|)$.¹⁰ We will denote a symmetric local public good economy by $E = \langle N; w; u; f \rangle$, with notations as in the previous sentences. We denote the set of symmetric local public good

¹⁰For any finite set J , $|J|$ denotes the number of elements of J .

economies in \mathcal{E} by \mathcal{ES} .

Let $E = \langle N; w; u; f \rangle \in \mathcal{ES}$. We define for each $k \in \mathbb{N}$ a level of local public good

$$y_k \in \arg \max_{y \geq 0} k u \left(w - \frac{1}{k} f(y, k), y, k \right).$$

This is a level of local public good that maximizes the total utility of the players in a jurisdiction of size k when the players in the jurisdiction share the costs of local public good production equally. Also, we define for each coalition of players $S \subseteq N$ a worth

$$v^E(S) = \max_{\mathbf{P} \in \mathcal{P}(S)} \sum_{J \in \mathbf{P}} \left(|J| u \left(w - \frac{1}{|J|} f(y_{|J|}, |J|), y_{|J|}, |J| \right) \right),$$

where $\mathcal{P}(S)$ denotes the set of all partitions of S . This is the maximum total utility obtainable by the players in coalition S when they consider forming various jurisdictions not including any players not in S and when the costs of local public good production are shared equally in each possible jurisdiction.

Note that the function v^E can be interpreted as the characteristic function of a coalition game with player set N . The game (N, v^E) is called *top convex* if $\frac{v^E(S)}{|S|} \leq \frac{v^E(N)}{|N|}$ for all $S \subseteq N$.

Theorem 2 shows that a share equilibrium exists for every symmetric local public good economy for which the associated coalitional game (N, v^E) is top convex.

Theorem 2. Let $E = \langle N; w; u; f \rangle \in \mathcal{ES}$ be a symmetric local public good economy such that the associated coalitional game (N, v^E) is top convex. Then $SE(E) \neq \emptyset$.

Proof. Define share indices \mathbf{s} by $s_i = 1$ for each $i \in N$, so that all players have the same share index. Also, let

$$P(N) \in \arg \max_{\mathbf{P} \in \mathcal{P}(N)} \sum_{J \in \mathbf{P}} \left(|J| u \left(w - \frac{1}{|J|} f(y_{|J|}, |J|), y_{|J|}, |J| \right) \right).$$

For each jurisdiction $J \in P(N)$, choose a

$$y_J \in \arg \max_{y \geq 0} |J|u \left(w - \frac{1}{|J|} f(y, |J|), y, |J| \right).$$

For each $i \in N$, denote the jurisdiction in $P(N)$ containing player i by $J(i)$ and define a level of private good consumption by

$$x_i = w - \frac{1}{|J(i)|} f(y_{J(i)}, |J(i)|).$$

We will prove that the share indices \mathbf{s} and the configuration $((x_i)_{i \in N}, (y_J)_{J \in P(N)}, P(N))$ form a share equilibrium in the symmetric economy E .

Claim 1. $\sum_{J \in P(N)} v^E(J) = v^E(N)$. This follows from the following string of (in)equalities.

$$\begin{aligned} \sum_{J \in P(N)} v^E(J) &= \sum_{J \in P(N)} \left(\max_{\mathbf{P} \in \mathcal{P}(J)} \sum_{K \in \mathbf{P}} \left(|K|u \left(w - \frac{1}{|K|} f(y_{|K|}, |K|), y_{|K|}, |K| \right) \right) \right) \\ &\leq \max_{\mathbf{P} \in \mathcal{P}(N)} \sum_{K \in \mathbf{P}} \left(|K|u \left(w - \frac{1}{|K|} f(y_{|K|}, |K|), y_{|K|}, |K| \right) \right) \\ &= v^E(N) \\ &= \sum_{J \in P(N)} \left(|J|u \left(w - \frac{1}{|J|} f(y_{|J|}, |J|), y_{|J|}, |J| \right) \right) \\ &\leq \sum_{J \in P(N)} \left(\max_{\mathbf{P} \in \mathcal{P}(J)} \sum_{K \in \mathbf{P}} \left(|K|u \left(w - \frac{1}{|K|} f(y_{|K|}, |K|), y_{|K|}, |K| \right) \right) \right) \\ &= \sum_{J \in P(N)} v^E(J) \end{aligned}$$

Claim 2. $v^E(J) = |J|u \left(w - \frac{1}{|J|} f(y_{|J|}, |J|), y_{|J|}, |J| \right)$ for all $J \in P(N)$. This follows from the fact that the last (weak) inequality in the sequence of (in)equalities above is an equality, as we have just derived.

Claim 3. $\frac{v^E(J)}{|J|} = \frac{v^E(N)}{|N|}$ for all $J \in P(N)$. To see this, we derive

$$v^E(N) = \sum_{i \in N} \frac{v^E(N)}{|N|} = \sum_{J \in P(N)} \sum_{i \in J} \frac{v^E(N)}{|N|} = \sum_{J \in P(N)} |J| \frac{v^E(N)}{|N|} \geq \sum_{J \in P(N)} v^E(J),$$

where the last inequality follows because top convexity of (N, v^E) implies that $|J| \frac{v^E(N)}{|N|} \geq v^E(J)$ for each $J \in P(N)$. Because $\sum_{J \in P(N)} v^E(J) = v^E(N)$ by Claim 1, it follows that all the weak inequalities are in fact equalities, so that we can derive that $|J| \frac{v^E(N)}{|N|} = v^E(J)$ for each $J \in P(N)$.

We are now ready to prove that $(\mathbf{s}, ((x_i)_{i \in N}, (y_J)_{J \in P(N)}, P(N)))$ is a share equilibrium. First note that for each potential jurisdiction the costs of local public good production are shared equally among all jurisdiction members because all players participate in the decision-making process and have the same share index. Condition 1 of the share equilibrium then follows from $x_i = w - \frac{1}{|J(i)|} f(y_{J(i)}, |J(i)|)$. To show that condition 2 of the share equilibrium also holds, fix $i \in N$ and let $(\bar{x}_i, \bar{y}, \bar{J}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times 2^N$ be such that $i \in \bar{J}$ and $\frac{1}{|\bar{J}|} f(\bar{y}, |\bar{J}|) + \bar{x}_i \leq w$. We need to show that $u_i(x_i, y_{J(i)}, J(i)) \geq u_i(\bar{x}_i, \bar{y}, \bar{J})$.

$$\begin{aligned}
u_i(\bar{x}_i, \bar{y}, \bar{J}) &\leq u\left(w - \frac{1}{|\bar{J}|} f(\bar{y}, \bar{J}), \bar{y}, |\bar{J}|\right) \\
&\leq \frac{1}{|\bar{J}|} |\bar{J}| u\left(w - \frac{1}{|\bar{J}|} f(\bar{y}_{|\bar{J}|}, \bar{J}), \bar{y}_{|\bar{J}|}, |\bar{J}|\right) \\
&\leq \frac{1}{|\bar{J}|} v^E(\bar{J}) \\
&\leq \frac{v^E(N)}{|N|} \\
&= \frac{v^E(J(i))}{|J(i)|} \\
&= u\left(w - \frac{1}{|J(i)|} f(y_{|J(i)|}, |J(i)|), y_{|J(i)|}, |J(i)|\right) \\
&= u_i(x_i, y_{J(i)}, J(i)),
\end{aligned}$$

where the first inequality follows from utility being strictly increasing in private good consumption, the second inequality follows from the definition of $\bar{y}_{|\bar{J}|}$, the third inequality follows from the definition of $v^E(\bar{J})$, the fourth inequality follows from top convexity of (N, v^E) , the first equality follows

by Claim 3, the second equality follows from Claim 2, and the last equality follows from the definitions of x_i and $y_{J(i)}$. ■

Obviously, the conditions given in the statement of Theorem 2 for existence of a share equilibrium are restrictive. The extent to which the conditions can be relaxed is the subject of ongoing research.

6 An axiomatization of the share equilibrium

It is apparent that SE is a special case of a mapping ϕ that assigns to each public good economy $E \in \mathcal{E}$ a set of pairs each consisting of a vector of numbers and a configuration, that is,

$$\phi(E) \subseteq \mathbb{R}^{D(E)} \times C(N(E)).$$

We will call such a mapping a *solution* on \mathcal{E} . We consider various properties of such solutions and show that these properties axiomatically characterize the share equilibrium. At the heart of our axiomatizations is the notion of consistency.

6.1 Consistency

Consistency dictates that, when the same method of reaching agreements is used in all groups, agreements reached in subgroups of players should be the same as those reached in the group consisting of all players. Suppose that the players in a local public good economy agree on their share indices, the jurisdictions formed, and the levels of local public good for these jurisdictions. The agreements are consistent if no subgroup R of players has an incentive to change its part of the agreement while taking the share indices of the

players in $N \setminus R$ as given. This means that the players in R have no incentive to change their own share indices, their jurisdictions of membership, or levels of local public good for those jurisdictions. Notice that, in interpretation, the players in $N \setminus R$ do not leave the economy but only the decision-making process. They still are present in their jurisdictions of membership and shoulder their previously agreed-upon shares of costs of local public good production in their assigned jurisdictions. As noted previously, modeling this situation motivated our introduction of a set of decision-making players and to simultaneously allow for the presence of non decision-making players.

The introduction of a distinction between decision-making players and non decision-making players, as noted previously, represents a new element in the standard local public good model. We need this additional dimension in order to be able to study reduced economies. The reason is that in a local public good economy, players have preferences not only over the level of public good provided in their jurisdiction, but also over whom they are sharing access to the public good with, i.e., their fellow jurisdiction members. Hence, when we consider agreements reached by a subgroup of players, we have to take into account the presence and contributions of players in the economy who are not involved in shaping the agreement. These non-participating players, however, will be bound by the agreement just the same and they have an obligation to shoulder their share of the cost of public good provision.

There are many situations in which people are obliged to pay their share of the cost of a public good according to some sharing formula even though they do not participate in the decision-making process. One of the authors recently considered purchasing a townhouse in a complex where the commons and exterior of the units are maintained by a home owners association. The covenants governing the home owners association, by which each owner of a unit in the complex is legally bound, stipulate that during any meeting

of the members at which at least 50% of all the votes in the association are represented, either is person or by proxy, a majority vote of those represented is binding on all members. Among other things, the members can decide to take on new projects and to increase members' contributions to the home owners association to cover the projects' costs. The covenants also stipulate that any such contributions have to be paid by each member within 30 days and (financial) penalties are spelled out for those not paying within 30 days.

Other examples of instances where people are obliged to pay their share of the cost of a public good according to some sharing formula even though they do not participate in the decision-making process, include parents of school-aged children who are not a member of the School Board and library members who do not participate in the management of the library.

Another way of looking at the issue of reducing a local public good economy is to ask what exactly it is that is fixed and what is subject to change when a group of players leaves the decision-making process. In a private good economy, players can claim ownership of their allotments of goods and it is natural to keep these allotments fixed for departing players. In a public good economy, however, that is obviously not the case as the public good is not owned by any one player. What is personalized in a public good economy is the cost shares of the players, whereas the jurisdiction memberships and levels of local public good are determined by collective pressures in much the same way as prices are in private good economies. Hence, where it is natural to introduce a net-trade obligation with outside members when reducing a private good economy (as in van den Nouweland et al. (2002)), it is natural to introduce the presence of non-decision making players when reducing a local public good economy.

We now formally introduce reduced economies. Take as given a local public good economy $E = \langle N, D; (w_i)_{i \in N}; (u_i)_{i \in N}; f_D \rangle$, let $R \subseteq D$ be a subset

of decision makers, $R \neq \emptyset$,¹¹ and let $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in \mathbb{R}^{D(E)} \times C(N(E))$. The *reduced economy* of E with respect to R and $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ is the economy in which the set of decision-making players is R , so

$$E^{R,(\mathbf{s},(\mathbf{x},\mathbf{y},\mathbf{P}))} = \langle N, R; (w_i)_{i \in N}; (u_i)_{i \in N}, f_R \rangle,$$

and where the production function f_R satisfies

$$f_R(\bar{y}, J) = \left[\sum_{i \in J \cap R} s_i^{J,D} \right] f_D(\bar{y}, J)$$

for all $\bar{y} \in \mathbb{R}_+$ and $J \subseteq N$.

The idea behind the definition of the reduced economy is as follows. Suppose all decision-making players agree on the share indices \mathbf{s} and configuration $(\mathbf{x}, \mathbf{y}, \mathbf{P})$. This implies that they agree on cost-sharing schemes corresponding to the share indices \mathbf{s} , formation of local jurisdictions \mathbf{P} , and levels of local public good production for each of those jurisdictions. Then, if, in addition to the players in $N \setminus D$, the players in $D \setminus R$ also withdraw from the decision-making process, the players in R can reconsider the jurisdictions that they form, the levels of local public good to be produced in those jurisdictions, and their relative shares of the residual cost of producing the local public good. When they reconsider, they take into account that the players in $D \setminus R$ have agreed to their share indices and will pay their corresponding shares for the cost of local public good production in the jurisdiction that they end up in. Hence, when reconsidering the cost-sharing scheme, the players in R face the residual cost $f_R(\bar{y}, J) = \left[1 - \sum_{i \in J \cap (D \setminus R)} s_i^{J,D} \right] f_D(\bar{y}, J) = \left[\sum_{i \in J \cap R} s_i^{J,D} \right] f_D(\bar{y}, J)$ for producing a level \bar{y} of local public good in a jurisdiction J .

A solution is consistent if, once agreement on relative cost shares has been reached, the withdrawal of some players from the decision-making process will

¹¹Think of R as standing for "reduced-economy decision-making agents".

not influence the final outcome of the process. The consistency property is defined using reduced economies. A solution ϕ on \mathcal{E} is *consistent* (CONS) if it satisfies the following condition.

$$\begin{aligned} &\text{If } E \in \mathcal{E}, (\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in \phi(E), \text{ and } R \subset D(E), \quad R \neq \emptyset, \\ &\text{then } E^{R, (\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))} \in \mathcal{E} \text{ and } (\mathbf{s}^R, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in \phi(E^{R, (\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))}). \end{aligned}$$

Here, $\mathbf{s}^R = (s_i)_{i \in R}$ denotes the vector of share indices of the members of R .

We illustrate the role of the share indices in consistency in the following example.

Example 1. Suppose that we have an economy with 3 players in which all players are decision-makers; $N = D = \{1, 2, 3\}$. Suppose the players agree on share indices $s_1 = s_2 = 1$ and $s_3 = 2$. Then, if jurisdiction $J_1 = \{1, 2\}$ is formed and a level of local public good y is produced in this jurisdiction, player 1's cost share would be $s_1^{J_1, D} = \frac{1}{2}$, so that player 1 would pay $\frac{1}{2}f_D(y, J_1)$. In jurisdiction $J_2 = \{1, 3\}$, player 1's cost share would be $s_1^{J_2, D} = \frac{1}{3}$ and he would pay $\frac{1}{3}f_D(y, J_2)$, and in jurisdiction $J_3 = N$, player 1 would pay $\frac{1}{4}f_D(y, J_3)$.

Now, suppose that player 1 leaves the decision-making process, agreeing to his share index $s_1 = 1$. In the reduced economy, the set of decision-making players is $R = \{2, 3\}$. The cost function $f_R(\cdot, \cdot)$ for the reduced economy is different from the cost function $f_D(\cdot, \cdot)$ only for jurisdictions that include player 1. Specifically, $f_R(y, J_1) = \frac{1}{2}f_D(y, J_1)$, $f_R(y, J_2) = \frac{2}{3}f_D(y, J_2)$, and $f_R(y, J_3) = \frac{3}{4}f_D(y, J_3)$. The remaining players can now reconsider not only the jurisdictions that they want to form and the levels of local public good for those jurisdictions, but also their share indices. Suppose that player 3 changes his share index to $\tilde{s}_3 = 1$. This changes player 3's relative standing vis-a-vis player 2 and shifts costs from player 3 to player 2. For example, in

jurisdiction $J_4 = \{2, 3\}$, player 3 will now have to pay the cost $\frac{1}{2}f_R(y, J_4) = \frac{1}{2}f_D(y, J_4)$, whereas before he changed his share index, he would have had to pay $\frac{2}{3}f_R(y, J_4) = \frac{2}{3}f_D(y, J_4)$. However, note that changing his share index does not allow player 3 to put a larger share of the (cost) burden of producing local public good on the non-decision-making player 1, as player 1's share of the cost in various jurisdictions was agreed upon before he left the decision-making process. For example, if player 3 now wants to form jurisdiction J_2 with player 1, then player 3 will still have to shoulder the cost $f_R(y, J_2) = \frac{2}{3}f_D(y, J_2)$. In jurisdiction J_3 with all three players, players 2 and 3 pay $\frac{1}{2}f_R(y, J_3) = \frac{3}{8}f_D(y, J_3)$ each.

The share equilibrium is a consistent solution, as is shown in the following lemma.

Lemma 1. The share equilibrium on the family \mathcal{E} of local public good economies is consistent.

Proof. Let $E = \langle N, D; (w_i)_{i \in N}; (u_i)_{i \in N}; f_D \rangle \in \mathcal{E}$ be a local public good economy, let $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in SE(E)$, and let $R \subseteq D$, $R \neq \emptyset$. Let f_R be the cost function of the reduced economy $E^{R, (\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))}$; that is,

$$f_R(\bar{y}, J) = \left[\sum_{i \in J \cap R} s_i^{J, D} \right] f_D(\bar{y}, J) \geq 0$$

for every $\bar{y} \in \mathbb{R}_+$ and $J \subseteq N$. Obviously, $E^{R, (\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))} \in \mathcal{E}$.

For all $i \in R$ and $J \subseteq N$ with $i \in J$ it holds that

$$\begin{aligned} s_i^{J, D} f_D(\bar{y}, J) &= \frac{s_i}{\sum_{j \in J \cap D} s_j} f_D(\bar{y}, J) \\ &= \frac{s_i}{\sum_{j \in J \cap R} s_j} \frac{\sum_{j \in J \cap R} s_j}{\sum_{j \in J \cap D} s_j} f_D(\bar{y}, J) \\ &= \frac{s_i}{\sum_{j \in J \cap R} s_j} \left[\sum_{j \in J \cap R} s_j^{J, D} \right] f_D(\bar{y}, J) \\ &= \frac{s_i}{\sum_{j \in J \cap R} s_j} f_R(\bar{y}, J) = s_i^{J, R} f_R(\bar{y}, J) \end{aligned}$$

for all $\bar{y} \in \mathbb{R}_+$. We now derive

$$s_i^{P(i),R} f_R(y_{P(i)}, P(i)) + x_i = s_i^{P(i),D} f_D(y_{P(i)}, P(i)) + x_i = w_i.$$

Let $(\bar{x}_i, \bar{y}, \bar{J}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times 2^N$ be such that $i \in \bar{J}$ and $s_i^{\bar{J},R} f_R(\bar{y}, \bar{J}) + \bar{x}_i \leq w_i$. Then $s_i^{\bar{J},D} f_D(\bar{y}, \bar{J}) + \bar{x}_i \leq w_i$ and, because $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in SE(E)$, we know that $u_i(x_i, y_{P(i)}, P(i)) \geq u_i(\bar{x}_i, \bar{y}, \bar{J})$. This proves that $(\mathbf{s}^R, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in SE(E^{R,(\mathbf{s},(\mathbf{x},\mathbf{y},\mathbf{P}))})$. ■

6.2 An axiomatization using consistency

The share equilibrium for local public good economies can be axiomatized using the consistency property introduced in the previous subsection and two additional axioms – converse consistency and one-person rationality.

Whereas consistency states that agreements that are acceptable for the group of all players should be acceptable in all smaller groups as well, converse consistency states that if a set of share indices and a configuration constitute an acceptable solution for all proper subgroups of players, then they also constitute an acceptable solution for the group as a whole. Formally, a solution ϕ on \mathcal{E} is *converse consistent* (COCONS) if, for every $E \in \mathcal{E}$ with at least two players ($|N(E)| \geq 2$) and for every set of share indices $\mathbf{s} = (s_i)_{i \in D(E)} \in \mathbb{R}^{D(E)}$ and every configuration $(\mathbf{x}, \mathbf{y}, \mathbf{P}) = ((x_i)_{i \in N}, (y_P)_{P \in \mathbf{P}}, \mathbf{P})$, the following condition is satisfied.

If $E \in \mathcal{E}$ and for every $R \subset D(E)$ with $R \notin \{\emptyset, D(E)\}$ it holds that $E^{R,(\mathbf{s},(\mathbf{x},\mathbf{y},\mathbf{P}))} \in \mathcal{E}$ and $(\mathbf{s}^R, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in \phi(E^{R,(\mathbf{s},(\mathbf{x},\mathbf{y},\mathbf{P}))})$,
then $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in \phi(E)$.

This means that for an economy with 3 players who are all decision-makers, for example, if a set of share indices and a configuration induce a share equilibrium in all 1- and 2-player reduced economies, then they must form

a share equilibrium in the 3-player economy. Hence, we can determine if a vector of share indices and a configuration form a share equilibrium by checking the reduced economies.

It is shown in the following lemma that the share equilibrium satisfies converse consistency.

Lemma 2. The share equilibrium on the family \mathcal{E} of local public good economies satisfies converse consistency.

Proof. Let $E = \langle N, D; (w_i)_{i \in N}; (u_i)_{i \in N}; f_D \rangle \in \mathcal{E}$ be a local public good economy with $|N| \geq 2$ and let share indices $\mathbf{s} = (s_i)_{i \in D} \in \mathbb{R}^D$ and configuration $(\mathbf{x}, \mathbf{y}, \mathbf{P}) = ((x_i)_{i \in N}, (y_P)_{P \in \mathbf{P}}, \mathbf{P}) \in C(N)$ be such that, for every $R \subseteq D$ with $R \notin \{\emptyset, D\}$, it holds that $E^{R, (\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))} \in \mathcal{E}$ and $(\mathbf{s}^R, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in SE(E^{R, (\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))})$. Then $(\mathbf{s}^{\{i\}}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in SE(E^{\{i\}, (\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))})$ for each $i \in D$.

Let $i \in D$ and let $f_{\{i\}}$ be the cost function of the reduced economy $E^{\{i\}, (\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))}$; that is, $f_{\{i\}}(\bar{y}, J) = s_i^{J, D} f_D(\bar{y}, J)$ for all $\bar{y} \in \mathbb{R}_+$ and $J \subseteq N$ with $i \in J$. Note that $s_i^{J, \{i\}} = 1$ for all $J \subseteq N$ with $i \in J$. Applying conditions 1 and 2 of share equilibrium to $(\mathbf{s}^{\{i\}}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$, we find that $f_{\{i\}}(y_{P(i)}, P(i)) + x_i = w_i$ and $u_i(x_i, y_{P(i)}, P(i)) \geq u_i(\bar{x}_i, \bar{y}, \bar{J})$ for all $(\bar{x}_i, \bar{y}, \bar{J}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times 2^N$ satisfying $i \in \bar{J}$ and $f_{\{i\}}(\bar{y}, \bar{J}) + \bar{x}_i = w_i$. Using $f_{\{i\}}(\bar{y}, J) = s_i^{J, D} f_D(\bar{y}, J)$ for all $\bar{y} \in \mathbb{R}_+$ and $J \subseteq N$ with $i \in J$, we then see that conditions 1 and 2 of share equilibrium are satisfied when applied to $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ and player i .

Since we chose player $i \in D$ arbitrarily, we can conclude that conditions 1 and 2 of share equilibrium are satisfied for $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))$ and each $i \in D$, which proves that $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in SE(E)$. ■

Consistency and converse consistency link solutions in larger economies to those for smaller economies and vice versa. One-person rationality considers solutions with one decision-making player only. A solution ϕ on \mathcal{E} satisfies *one-person rationality* (OPR) if, for every local public good economy with

one decision-making player $E = \langle N, \{i\}; (w_j)_{j \in N}; (u_j)_{j \in N}; f_{\{i\}} \rangle \in \mathcal{E}$, it holds that

$$\begin{aligned} \phi(E) &= \{(s_i, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \mid s_i > 0, f_{\{i\}}(y_{P(i)}, P(i)) + x_i = w_i \\ &\text{and } u_i(x_i, y_{P(i)}, P(i)) \geq u_i(\bar{x}_i, \bar{y}, \bar{J}) \text{ for all } (\bar{x}_i, \bar{y}, \bar{J}) \\ &\text{satisfying } i \in \bar{J} \text{ and } f_{\{i\}}(\bar{y}, \bar{J}) + \bar{x}_i \leq w_i\}. \end{aligned}$$

The one-person rationality axiom dictates that in a one-decision-maker economy, the single decision-making player maximizes his utility given his endowment of the private good and the cost of producing certain amounts of local public good in various jurisdictions when the player has to pay all the residual cost of local public good provision. This is a rationality assumption much like those that prevail throughout economics.

The interaction of the three axioms consistency, converse consistency, and one-person rationality is explained in the following lemma.

Lemma 3. Let ϕ and ψ be two solutions on \mathcal{E} that both satisfy one-person rationality. If ϕ is consistent and ψ is converse consistent, then it holds that $\phi(E) \subseteq \psi(E)$ for all $E \in \mathcal{E}$.

Proof. We will prove the lemma by induction on the number of players.

If $E \in \mathcal{E}$ is an economy with one decision-making player - $|D(E)| = 1$ - then it follows from OPR of ϕ and ψ that $\phi(E) = \psi(E)$.

Now, let $E \in \mathcal{E}$ be an economy with n decision-making players and suppose that it has already been proven that $\phi(E) \subseteq \psi(E)$ for all economies with fewer than n decision-making players. Let $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in \phi(E)$. Then, by CONS of ϕ , we know that $E^{R, (\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))} \in \mathcal{E}$ and $(\mathbf{s}^R, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in \phi(E^{R, (\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))})$ for all $R \subseteq D(E)$, $R \notin \{\emptyset, D(E)\}$. Hence, it follows from the induction hypothesis that $(\mathbf{s}^R, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in \psi(E^{R, (\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P}))})$ for all $R \subseteq D(E)$, $R \notin \{\emptyset, D(E)\}$. So, by COCONS of ψ , we know that $(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \in \psi(E)$. We conclude that $\phi(E) \subseteq \psi(E)$. ■

Theorem 3 shows that consistency, converse consistency, and one-person rationality characterize the share equilibrium.

Theorem 3. The share equilibrium is the unique solution on \mathcal{E} that satisfies one-person rationality, consistency, and converse consistency.

Proof. In Lemmas 1 and 2 we proved that the share equilibrium satisfies CONS and COCONS. To show that the share equilibrium satisfies OPR, let $E = \langle N, \{i\}; (w_j)_{j \in N}; (u_j)_{j \in N}; f_{\{i\}} \rangle \in \mathcal{E}$ be a local public good economy with one decision-making player. Note that in an economy with one decision-making player, the single decision-making player present will have to pay fully the remaining cost for each level of local public good that he wants to have available in a jurisdiction. The players' share index plays no role in this case as there are no other decision-making players so that it needs not be determined how to share the cost of local public good production among decision-making players. Hence, the decision-making player simply determines for each possible jurisdiction the optimal level of local public good within his budget and chooses the jurisdiction and local public good combination that maximizes his utility. This is exactly what OPR describes.

To prove uniqueness, assume that ϕ is a solution on \mathcal{E} that also satisfies the three foregoing axioms. Let $E \in \mathcal{E}$ be arbitrary. Then, Lemma 3 shows that $\phi(E) \subseteq SE(E)$ by CONS of ϕ and COCONS of the share equilibrium, and that $SE(E) \subseteq \phi(E)$ by CONS of the share equilibrium and COCONS of ϕ . Hence, $\phi(E) = SE(E)$. ■

We conclude this section with the remark that the three axioms used to characterize the share equilibrium in Theorem 3 are logically independent. This is easily seen by considering the following three solutions on \mathcal{E} . First, consider a solution ϕ on \mathcal{E} that gives each decision-making player the same share index, assigns players to jurisdictions arbitrarily, has a level of local

public good equal to 0 for each jurisdiction formed, and let each player consume his entire initial endowment (so that no player pays for local public good provision). Define ϕ by $\phi(E) = \{(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \mid s_i = 1 \text{ and } x_i = w_i \text{ for each } i \in N(E), \mathbf{P} \text{ is a partition of } N(E), \text{ and } y_P = 0 \text{ for all } P \in \mathbf{P}\}$. This solution satisfies CONS and COCONS, but fails to satisfy OPR. Second, consider the solution χ on \mathcal{E} that coincides with the share equilibrium for economies with one decision-making player, and is empty for economies with more than one decision-making player. Formally, χ is defined by $\chi(E) = SE(E)$ if $|D(E)| = 1$ and $\chi(E) = \emptyset$ if $|D(E)| > 1$. This solution satisfies OPR and CONS, but does not satisfy COCONS. Finally, consider the solution ψ on \mathcal{E} that coincides with the share equilibrium for economies with one decision-making player, and for economies with more than one decision-making player assigns arbitrary share indices to players, groups them into jurisdictions arbitrarily, and has levels of local public good and of private-good consumption that are such that in each jurisdiction formed its members pay for the public good provided. So, ψ is defined by $\psi(E) = SE(E)$ if $|D(E)| = 1$ and $\psi(E) = \{(\mathbf{s}, (\mathbf{x}, \mathbf{y}, \mathbf{P})) \mid \mathbf{s} \in \mathbb{R}_{++}^{D(E)}, \mathbf{P} \text{ is a partition of } N(E), x_i \leq w_i \text{ for all } i \in N(E) \text{ and } \sum_{i \in P \cap D} (w_i - x_i) = f_D(y_P, P) \text{ for each } P \in \mathbf{P}\}$ if $|D(E)| > 1$. This solution satisfies OPR and COCONS, but does not satisfy CONS.

7 Conclusions

To summarize, in this paper, we introduced the concept of a share equilibrium for local public good economies where relative share indices of players determine their shares of costs of local public good provision in each possible jurisdiction to which they may belong. The relative cost shares, and hence the relative share indices of players who end up in the same jurisdiction, can be determined just by considering the differences between their initial

endowment and their consumption of private good. Note, however, that the share indices also contain information on relative cost shares of players in *hypothetical* jurisdictions, that is, ones that do not appear in the equilibrium jurisdiction structure. In pure public good economies, where all players are by definition in the same jurisdiction, there is no need to consider hypothetical jurisdictions and the cost shares of players in such jurisdictions. Hence, for pure public good economies, we can suffice by specifying players' actual cost shares, or their relative share in the unique jurisdiction. In this manner, we obtain the ratio equilibrium for pure public good economies, as defined in Kaneko (1977a,b) and studied in van den Nouweland, Tijs, and Wooders (2002), as a special case of the share equilibrium. If all players have the same share index the equal cost sharing of Konishi, Le Breton and Weber (1998), for example, is another special case.

Since a share equilibrium is a new concept, there are many open questions. In the current paper, we show that a share equilibrium leads to jurisdictions and consumption levels of local public good and private good that are in the core of an economy and we prove a existence of share equilibria for a class of symmetric local public good economies. We also provide an axiomatic characterization of the share equilibrium. This gives us insight into the properties of this equilibrium concept and shows that it is in the same spirit as the ratio equilibrium. In ongoing research we continue to address questions on existence of share equilibrium and the relationship between share equilibrium and Lindahl equilibrium, as well as other questions relating to share equilibrium.

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