Quantum Chaos: From Electron Waves to Rogue Ocean Waves to Microwaves

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Wave motion through weak correlated disorder:

- Electron flow in nanostructures (10^{-6} m)
- Rogue waves on the ocean (10^{5} m)
- Microwaves in disordered cavities (10^{-1} m)

Brings together several themes of Heller’s work:

- Wave dynamics
- Chaos
- Scattering
- Semiclassical methods
I. Electron Flow in Nanostructures

- Motivated by experiments by Topinka et al. (2001)
- Electrons confined to 2-dimensional electron gas (2DEG) inside GaAs-AlGaAs heterostructure
- Gate voltages used to create barriers inside 2DEG and carve out region through which electrons may flow
I. Electron Flow in Nanostructures

- Barriers chosen so electrons pass through quantum point contact (QPC)
- Scanning probe microscopy used to image electron flow through such a 2DEG device
  - Negatively charged tip reflects current and reduces conductance through device
  - By measuring reduction in conductance as function of tip position, can map out regions of high and low current
Application I: Electron Flow in Nanostructures

- Known: potential $V(x,y)$ away from gates is not zero, but is random due to donor ions and impurities in 3d bulk
- Expected: outgoing flow should exhibit random wave pattern (Gaussian random amplitude fluctuations)
- Instead: observed strong current concentrated in small number of “branches” (Topinka et al., Nature 2001)
Understanding branching of electron flow

- Simplest description of random potential: Gaussian random with rms height $V_{\text{RMS}}$ and correlation distance $d$
- Numerical simulations of the Schrödinger equation show qualitatively similar branching behavior
- Also seen in classical simulation!
  - Semiclassical or ray picture must be applicable
  - Branches do not correspond to “valleys” of the potential
Understanding branching of electron flow

- Consider parallel incoming paths encountering single shallow dip in potential energy $V(x,y)$
- **Focusing** when all paths in a given neighborhood coalesce at a single point (caustic), producing infinite ray density
- Different groups of paths coalesce at different points ("bad lens" analogy)
Understanding branching of electron flow

- Generic result: “cusp” singularity followed by two lines of “fold” caustics
- At each $y$ after cusp singularity, we have infinite density at some values of $x$

Phase space picture
Refraction by weak, random potential

- Small angle scattering
- First singularities form after
- Further evolution: exponential proliferation of caustics
  - Tendrils decorate original branches
  - Branch statistics described by single distance scale $D$
More artistic visualization of electron flow

(Eric Heller)
Computing statistics of branched wave flow
[LK, PRL 2002]

- Singularities washed out on wavelength scale
  - Visible branch only if smeared intensity above background

- As we move away from QPC:
  - # of caustics grows exponentially
  - Typical intensity of each caustic decays exponentially due to stretching of phase space manifold

- But not all pieces of manifold stretch at same rate
  - Visible branch occurs only when singularity occurs in piece of manifold that had stretched anomalously little
Computing statistics of branched wave flow

- When all distances expressed in terms of D, everything governed by 2 dimensionless numbers (describing log-normal distribution of stretching factors)
  - $\alpha$: average rate of stretching
  - $\beta$: variance of stretching rate
- Then # of visible branches decays exponentially as
  \[ \exp[-(\alpha^2/\beta) \, y/D] \]
- Intensity of strongest branch:
  \[ \ln(I_{\text{max}}) = \frac{1}{2} \ln(d/\lambda) - \gamma (y/D) \]
  where \[ \gamma = \alpha - \frac{\beta}{2} \left[ \sqrt{1 + 4 \frac{\alpha}{\beta}} - 1 \right] \]
  and $\lambda$ is the wavelength
Numerical simulations

- Intensity of strongest branch:

![Graph showing intensity of strongest branch with a legend indicating various parameters and a theory line.](image)
Numerical simulations

- Fraction of space covered by visible branches:

- Also, compute distribution of branch heights, etc.
II. Rogue Waves
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- ~ 10 large ships lost per year to presumed rogue waves – usually no communication
- Also major risk for oil platforms in North Sea, etc.
- Probability greatly exceeds random wave model predictions (=Rayleigh distribution)
- Height up to 34 m, last for minutes or hours
- Long disbelieved by oceanographers, first hard evidence in 1995 (North Sea)
- Tend to form in regions of strong current: Agulhas, Kuroshio, Gulf Stream
Rogue Waves: Refraction Picture

- Various explanations: nonlinear instabilities ...

- Here, focus on linear refraction of incoming wave (velocity $v$) by random current eddies (typical current speed $u_{\text{rms}} \ll v$)

- For deep water surface gravity waves with current

$$
\frac{dk_x}{dt} = - \frac{\partial \omega}{\partial x}, \quad \frac{dx}{dt} = \frac{\partial \omega}{\partial k_x}, \quad \omega(\mathbf{r}, \mathbf{k}) = \sqrt{gk + \bar{u}(\mathbf{r}) \cdot \mathbf{k}}
$$

- Current correlated on scale $\xi$ (typical eddy size)

- Ray picture justified if $k\xi \gg 1$
Rogue Waves: Key Ideas

- Here, initial spread of wave directions important
  - New parameter $\Delta k (= k \Delta \theta)$
- Causes smearing of singularities
- “Hot spots” of enhanced average energy density remain as reminders of where caustics would have been

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Quantifying residual effect of caustics: the “freak index”

- Define freak index $\gamma = \delta k_x / \Delta k_x$

- Equivalently $\gamma = \delta \theta / \Delta \theta$
  - $\Delta \theta = \Delta k_x / k = \text{initial directional spread}$
  - $\delta \theta = \text{typical deflection before formation of first cusp}$
  - $\sim (u_{\text{rms}} / v)^{2/3} \sim \xi / D$

- Most dangerous: well-collimated sea impinging on strong random current field ($\gamma \geq 1$)

- Hot spots corresponding to first smooth cusps have highest energy density

\[
\Delta \theta(y) \approx \Delta \theta \sqrt{1 + \gamma^2 (y / D)} \quad \Rightarrow \quad \gamma(y) \approx \sqrt{D / y}
\]
Typical ray calculation for ocean waves

\[ \Delta \theta = 5^\circ \ (\gamma=3.5) \]

\[ \Delta \theta = 25^\circ \ (\gamma=0.7) \]
Implications for Freak Wave Statistics

- Simulations (using Schrodinger equation): long-time average and regions of extreme events

1 SWH = significant wave height $\approx 4\sigma$ crest to trough

$\gamma \approx 2$
Calculation: Freak Waves

No currents: Rayleigh height distribution (random superposition of waves)

\[ P(h) = \left(\frac{h}{\sigma}\right) \exp\left(-\frac{h^2}{2\sigma^2}\right) \]

With currents: Superpose locally Gaussian wave statistics on pattern of “hot/cold spots” caused by refraction

- Local height distribution

\[ P_l(h) = \left(\frac{h}{\sqrt{I}\sigma}\right) \exp\left(-\frac{h^2}{2I\sigma^2}\right) \]

- \( I(\vec{r}) \) is position-dependent variance of the water elevation (proportional to ray density; high in focusing regions, low in defocusing regions)

- Total wave height distribution:

\[ P(h) = \int P_l(h) f(I) dI \]
Thus, calculate wave height probability by combining

- Rayleigh distribution
  \[ P(h) = \left(\frac{h}{\sigma}\right) \exp\left(-\frac{h^2}{2\sigma^2}\right) \]

- Distribution \( f(I) \), which describes ray dynamics and depends on scattering strength (freak index)

Rogue wave forecasting??
Analytics: limit of small freak index

For $\gamma << 1$: $f(I)$ well approximated by $\chi^2$ distribution of $n \sim \gamma^{-2}$ degrees of freedom (mean 1, width $\sim \gamma$)

$$P(h > x \cdot \text{SWH}) = \int \exp\left(-\frac{2x^2}{I}\right) f(I) \, dI$$

$$P(h > x \cdot \text{SWH}) = \frac{2(nx^2)^{n/4}}{\Gamma(n/2)} K_{n/2}(2\sqrt{nx})$$
Analytics: limit of small freak index

Perturbative limit: for $x^2 \gamma << 1$ ($x << n^{1/4}$)

$$P(h > x \cdot SWH) = \frac{2(nx^2)^{n/4}}{\Gamma(n/2)} K_{n/2}(2\sqrt{nx})$$

Asymptotic limit: for $x \gamma^3 >> 1$ ($x >> n^{3/2}$)

$$P(h > x \cdot SWH) = \left[ 1 + \frac{4}{n} (x^4 - x^2) \right] \exp(-2x^2)$$

$$P(h > x \cdot SWH) = \frac{\sqrt{\pi} (\sqrt{nx})^{(n-1)/2}}{\Gamma(n/2)} \exp(-2\sqrt{nx})$$
Wave height distribution for ocean waves

$\gamma = 3.5$

$\gamma = 2.8$

$\gamma = 2.1$

$\gamma = 1.4$

$\gamma = 0.7$
Extension to nonlinear waves (NLSE)

Current work by Linghang Ying

- Again, full distribution well approximated by convolution of Rayleigh (random waves) with $\chi^2$ distribution of $n$ degrees of freedom

\[
iA_t + \frac{1}{4}A_{xx} - \frac{1}{8}A_{yy} - \frac{1}{2}|A|^2 A - U_x A = 0
\]

where

\[
\Psi(x, y, t) \sim A(x, y, t)e^{ik_0x - i\omega_0 t}
\]

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Extension to nonlinear waves

- Wave height distribution again depends on single parameter $n$ (Rayleigh as $n \to \infty$)

- How does $n$ depend on wave steepness, incoming angular spread, spectral width, etc?
Extension to nonlinear waves

$n$ as function of incoming angular spread for fixed steepness and frequency spread

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Extension to nonlinear waves

$n$ as function of incoming frequency spread for fixed steepness
Extension to nonlinear waves

$n$ as function of steepness for fixed incoming angular spread and frequency spread

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III. Microwave scattering experiments

Quasi-2D resonator with randomly placed scatterers
(With R. Höhmann, U. Kuhl, and H-J. Stöckmann)

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Microwave scattering experiments

Experimentally observed branched flow

Ray simulation
Microwave scattering experiments

Extreme event

Time Evolution

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Microwave scattering

Wave intensity in time and space
Rogue waves

Monsters of the deep
Sep 17th 2009
From The Economist print edition

Huge, freak waves may not be as rare as once thought

ON JULY 26th 1909 the SS Waratah, en route to London from Melbourne, left Durban with 211 passengers and crew. She was due in Cape Town three days later but never arrived. The steamship was last sighted along the east coast of South Africa—known to sailors as the “wild coast” for its violent weather—struggling through a stormy sea with waves more than nine metres (30 feet) high. No trace of the vessel has ever been found.

A theory which might explain her disappearance, and that of some other vessels, is that they were struck by rogue waves—which begin with a deep trough followed by a wall of water the size of an eight- or nine-storey building. For many years oceanographers dismissed sailors’ reports of rogue waves much as they did stories of mermaids. But in 1995 an oil rig in the North Sea recorded a 25.6-metre wave. Then in 2000 a British oceanographic vessel recorded a 29-metre wave off the coast of Scotland. In 2004 scientists using three weeks of radar images from European Space Agency satellites found ten rogue waves, each 25 metres or more high.

A typical ocean wave forms when wind produces a ripple across the surface of the sea. If the wind is strong, the ripples grow larger. A hurricane can amplify a wave to a few storeys. But trying to create giant rogue waves in a laboratory tank is very difficult, making them hard to study. Now researchers led by Eric Heller of Harvard University and Lev Kaplan of Tulane University, New Orleans, have started using microwaves rather than water waves to create a
Rogue waves

Monsters of the deep

Huge, freak waves may not be as rare as once thought

ON JULY 26th 1999 the SS Warriah, on a voyage from London to Melbourne, left Durban with 20 passengers and crew. She was due in Cape Town three days later but never arrived. The steamship was lost sighted along the east coast of South Africa—known to sailors as the “wild coast” for its violent weather—sinking through a stormy sea with waves more than nine metres (30 feet) high. No trace of the vessel has ever been found.

A theory which might explain her disappearance, and that of some other vessels, is that they were struck by rogue waves—which begin with a deep trough followed by a wall of water the size of an eight-storey building. For many years oceanographers dismissed sailors' reports of rogue waves much as they did stories of mermaids. But in 1995 an outbreak in the North Sea recorded a 25-metre wave. Then in 2002 a British oceanographic vessel recorded a 29-metre wave off the coast of Scotland. In 2004 scientists using three weeks of radar images from European Space Agency satellites found ten rogue waves, each 25 metres or more high.

A typical ocean wave forms when wind produces a ripple across the surface of the sea. If the wind is strong, the ripples are set in motion by earthquakes. These travel at high speed, building up as they approach the shore. Rogue waves seem to occur in deep water or where a number of physical factors such as strong winds and fast currents converge. This may have a focusing effect, which causes a number of waves to join together. Such conditions exist along Africa's wild coast, where strong winds blowing from the north west interact with the swell and narrow Agulhas current flowing down the coast to produce enormous waves. Dr Heller, who likes to sail, says there may be other mechanisms at work too, including an interference effect that creates different ocean swells, travelling at different speeds, to add up to produce a rogue, and a non-linear effect in which a small change in something like wind direction or speed causes a disproportionately large wave.

To study the phenomenon the group created a platform measuring 20cm by 35cm on which they randomly placed around 60 small brass cones to mimic random eddies in ocean currents. When microwave beams were directed at the platform, the researchers found that hot spots (the microwave equivalent of rogue waves) appeared far more often than conventional wave theory would predict; they were between ten and 100 times more likely.

Dr Heller says the results tend to support anecdotal evidence from sailors that rogue waves are not as rare as once thought. He thinks the work could also be used to understand more about the formation of these dangerous waves, perhaps in the point where it would one day be possible to provide a warning in places where rogue waves may prove to be greater. Seafarers would be thankful for that.

The Economist September 5th 2009

Teenage sexual maturity

Daddy's girl

A non-obvious explanation for why girls without fathers have sex earlier

It has long been a puzzle that girls who grow up without their fathers at home reach sexual maturity earlier than girls whose fathers live with them. For years, absent fathers have taken the blame for a reproductive strategy to their circumstances. If life is harsh, the theory goes, maybe they need to get their babies into the world as quickly as possible.

The animal world suggests that this effect is not restricted to humans. Young female mice, pigs, goats and even a few primates get signals from their kin which inhibit sexual development: a strange male in their midst, by contrast, really speeds things up. Research in humans has shown that girls growing up with stepfathers mature even more quickly than fatherless girls and that stepmothers have a measurable effect too.

However, Jan Mendle of the University of Oregon and her colleagues have suggested another putative cause: germs. Specifically, the same germ that might make a dad more likely to leave his family could be behind early sexual development as well.

The researchers came to their conclusion after analysing data collected through the American National Longitudinal Survey of Youth. Dr Mendle looked at 1,382 boys and girls, each of whom was related to at least one other subject through their mother. Most of the mothers were pairs of sisters, but some were identical twins or first cousins raised as sisters.

The study included questions about many things, including whether the father of their children lived with them. They were surveyed every year from 1979 to 1994 and then every second year. From age 14 their children were given annual questionnaires, and asked if they had engaged in sexual intercourse yet.

What the researchers wanted to know was whether the age at which a young person first had sex was something that ran in the family—regardless of whether the father had been around or not. To find out, they compared young people who had grown up without their dads with others whose dads remained at home. If the environmental effect of a father’s absence was causing children, to mature faster, they reasoned, that would show up.

It didn’t. In fact, the more closely related the twins were—by having mothers who were identical twins, for instance, versus cousins—the closer a child’s age at first sexual experience, says Dr Mendle. The researchers found it was just as true for boys as it was for girls. They published their work in the current issue of Child Development.

Dr Mendle now suspects that the same genetic factors that influence when a child first has intercourse also affect the type of the relationship.
Summary

- Branched Flow in Nanostructures
  - Simple scaling behavior in semiclassical limit
  - Branch statistics can be computed analytically
  - Generalizes to any weak semiclassical scattering

- Rogue Waves in the Ocean
  - Refraction of random incoming sea by random currents produces lumpy energy density
  - Skews formerly Rayleigh distribution of wave heights
  - Importance of refraction quantified by freak index $\gamma$
  - Spectacular effects in tail even for small $\gamma$
  - Goal: combine refractive effects with nonlinearity
Summary

- Microwave Cavity Experiments
  - Random scatterers play role of current eddies
  - Confirm that linear wave dynamics (refraction) sufficient to produce rogue wave phenomenon
  - Great laboratory for comparing analytic results with experiment