Linear and Nonlinear Rogue Wave Statistics in the Presence of Random Currents

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October 11, 2011
Talk Outline

- Linear rogue wave formation
  - Mechanism: Refraction from current eddies
  - Combining refraction with stochasticity
  - The “freak index”
  - Implications for extreme wave statistics
- Introducing wave nonlinearity
  - Wave statistics in 4\textsuperscript{th} order nonlinear equation
  - Scaling with parameters describing sea state
- Can linear and nonlinear effects work in concert?
Focus on linear **refraction** of incoming wave (velocity \( \nu \)) by **random current eddies** (current speed \( u_{\text{rms}} \ll \nu \))

- For deep water surface gravity waves with current

\[
\omega(\vec{r}, \vec{k}) = \sqrt{gk + \ddot{u}(\vec{r}) \cdot \vec{k}}
\]

\[
\frac{dk_x}{dt} = - \frac{\partial \omega}{\partial x}
\]

\[
\frac{dx}{dt} = \frac{\partial \omega}{\partial k_x}
\]

- Current correlated on scale \( \xi \) (typical eddy size)

- Ray picture justified if \( k\xi \gg l \)
How caustics form: refraction from current eddies

- Parallel incoming rays encountering pair of eddies
- Effect similar to potential dip in particle mechanics
- **Focusing** when all paths in a given neighborhood coalesce at a single point (caustic), producing infinite ray density
- Different groups of paths coalesce at different points ("bad lens" analogy)
How caustics form: refraction from current eddies

- Generic result: “cusp” singularity followed by two lines of “fold” caustics
- At each $y$ after cusp singularity, we have infinite density at some values of $x$

Phase space picture
Refraction from weak, random currents

- First singularities form after \( d \propto \xi \left( \frac{u_{\text{rms}}}{v} \right)^{-2/3} \gg \xi \)
- Further evolution: exponential proliferation of caustics
  - Tendrils decorate original branches
  - Branch statistics described by single distance scale \( d \)
Refraction from weak, random currents

- Of course, singularities washed out on wavelength scale

- Qualitative structure independent of
  - dispersion relation (e.g. $\omega \sim k^2$ for Schrödinger vs. $\omega \sim k^{1/2}$ for ocean waves)
  - details of random current field

- Can calculate distribution of branch strengths, fall off with distance, etc (LK, PRL 2002)
Analogies with other physical systems

- Electron flow in nanostructures (10^{-6} m)
- Microwave resonators, Stöckmann group (1 m)
- Long-range ocean acoustics, Tomsovic et al (20 km)
- Twinkling of starlight, Berry (2000 km)
- Gravitational lensing, Tyson (10^9 light years)

Microwave scattering experiments

Höhmann et al (PRL 2010)

Experimentally observed branched flow

Ray simulation
Branching pattern for tsunami waves
Problems with refraction picture of rogue waves

- Assumes single-wavelength and unidirectional initial conditions, which are unrealistic and unstable (Dysthe)
  - Singularities washed out only on wavelength scale
  - Predicts regular sequence of extreme waves *every time*
  - No predictions for actual wave heights or probabilities

- Solution: replace incoming plane wave with random initial spectrum
  - Finite range of wavelengths and directions
Smearing of caustics for stochastic incoming sea

- Singularities washed out by randomness in initial conditions: $\Delta k$
- “Hot spots” of enhanced average energy density remain as reminders of where caustics would have been
Smearing of caustics for stochastic incoming sea

Competing effects of focusing and initial stochasticity:

\[ \Delta k_x = \text{initial wave vector spread} \]

\[ \delta k_x = \text{wave vector change due to refraction} \]
Quantifying residual effect of caustics: the “freak index”

- Define freak index \( \gamma = \frac{\delta k_x}{\Delta k_x} \)
- Equivalently \( \gamma = \frac{\delta \theta}{\Delta \theta} \)
  - \( \Delta \theta = \frac{\Delta k_x}{k} = \) initial directional spread
  - \( \delta \theta = \) typical deflection before formation of first cusp
    \(~ (u_{\text{rms}} / v)^{2/3} \sim \xi / d \)
- Most dangerous: well-collimated sea impinging on strong random current field \( (\gamma \geq 1) \)
- Hot spots corresponding to \( \text{first} \) smooth cusps have highest energy density
  \[ \Delta \theta(y) \approx \Delta \theta \sqrt{1 + \gamma^2 (y / d)} \Rightarrow \gamma(y) \approx \sqrt{d / y} \]
Typical ray calculation for ocean waves

$\Delta \theta = 10^\circ$

$\Delta \theta = 20^\circ$
Implications for Rogue Wave Statistics

- Simulations (using Schrodinger equation): long-time average and regions of extreme events

Average

$\gamma \approx 2$

$1 \text{ SWH} = \text{significant wave height} \approx 4\sigma \text{ crest to trough}$

$>3 \text{ SWH}$

$>2.2 \text{ SWH}$
Calculation: Rogue Waves

No currents: Rayleigh height distribution (random superposition of waves) \( P(h) = \left(\frac{h}{\sigma}\right) \exp(-h^2 / 2\sigma^2) \)

With currents: Superpose *locally Gaussian* wave statistics on pattern of “hot/cold spots” caused by refraction

- Local height distribution \( P_l(h) = \left(\frac{h}{\sqrt{I}\sigma}\right) \exp(-h^2 / 2I\sigma^2) \)

- \( I(\vec{r}) \) is position-dependent variance of the water elevation (proportional to ray density; high in focusing regions, low in defocusing regions)

- Total wave height distribution: \( P(h) = \int P_l(h) f(I) dI \)
Thus, calculate wave height probability by combining

- Rayleigh distribution \( P(h) = \left(\frac{h}{\sigma}\right) \exp\left(-\frac{h^2}{2\sigma^2}\right) \)

- Distribution \( f(I) \), which describes ray dynamics and depends on scattering strength (freak index)

Rogue wave forecasting??
Analytics: limit of small freak index

\[ P(h > x \cdot SWH) = \int \exp\left( -2x^2 / I \right) f(I) \, dI \]

For \( \gamma << 1 \): \( f(I) \) well approximated by \( \chi^2 \) distribution of \( n \sim \gamma^{-2} \) degrees of freedom (mean 1, width \( \sim \gamma \))

\[ P(h > x \cdot SWH) = \frac{2(nx^2)^{n/4}}{\Gamma(n/2)} \cdot K_{n/2}(2\sqrt{nx}) \]
Analytics: limit of small freak index

\[ P(h > x \cdot SWH) = \frac{2(nx^2)^{n/4}}{\Gamma(n/2)} K_{n/2}(2\sqrt{nx}) \]

Perturbative limit: for \( x^2 \gamma << 1 \) (\( x << n^{1/4} \))

\[ P(h > x \cdot SWH) = \left[ 1 + \frac{4}{n} (x^4 - x^2) \right] \exp(-2x^2) \]

Asymptotic limit: for \( x \gamma^3 >> 1 \) (\( x >> n^{3/2} \))

\[ P(h > x \cdot SWH) = \frac{\sqrt{\pi} (\sqrt{nx})^{(n-1)/2}}{\Gamma(n/2)} \exp(-2\sqrt{nx}) \]
Numerical Simulations

- Incoming sea with $v=7.8 \, \text{m/s} \, (T=10 \, \text{s}, \, \lambda=156 \, \text{m})$
- Random current with $u_{\text{rms}} = 0.5 \, \text{m/s}$ and correlation $\xi=20 \, \text{km}$
- Dimensionless parameters:
  - $\lambda / \xi << 1$ (ray limit)
  - $\delta \theta \sim (u_{\text{rms}} / v)^{2/3} << 1$ (small-angle scattering)
  - $\Delta \theta = \text{spreading angle} = 5$ to $25^\circ$
  - $\gamma = \delta \theta / \Delta \theta = \text{freak index} \, (\gamma=3.5 \text{ to } 0.7)$
Wave height distribution for ocean waves

![Wave Height Distribution Graph]

- Numerical Data
- K Distribution
- Rayleigh

Wave Height / SWH

Cumulative Probability
Numerical test of $N$ vs $\gamma$
Probability enhancement over Rayleigh predictions

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<th>$\Delta \theta$</th>
<th>$\gamma$</th>
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$u_{\text{RMS}} = 0.5 \text{ m/s} \quad \nu = 7.8 \text{ m/s}$
Extension to nonlinear waves

NLS with current:

\[ iA_t + \frac{1}{4} A_{xx} - \frac{1}{8} A_{yy} - \frac{1}{2} |A|^2 A - U_x A = 0 \]

where

\[ \Psi(x, y, t) \sim A(x, y, t)e^{ik_0 x - i \omega_0 t} \]
Actually use 4th order equation (Stocker & Peregerine)

\[
iB_T - \frac{1}{8}(B_{XX} - 2B_{YY}) - \frac{1}{2}B|B|^2 - B\Phi_{cX} \\
= \frac{i}{16}(B_{XX} - 6B_{YY}) + \bar{\Phi}_X B + \frac{i}{4}B(BB^*_X - 6B^*_X) \\
+i(\frac{1}{2}\Phi_{cXT} - \Phi_{cZ})B - i(\Phi_{cX} B_X + \Phi_{cY} B_Y)
\]

Velocity potential: \( \phi = \sqrt{\frac{g}{k_0^2}} [\bar{\Phi} + \Phi_c + \frac{1}{2} (Be^{k_0z+i\theta} + B_2e^{2(k_0z+i\theta)} + \text{c.c.})] \)

Surface elevation: \( \zeta = k_0^{-1} [\bar{\zeta} + \zeta_c + \frac{1}{2} (A^{i\theta} + A_2e^{2i\theta} + A_3e^{3i\theta} + \text{c.c.})] \)

Relation between expansion coefficients:

\[
A = iB + \frac{1}{2k_0}B_x + \frac{i}{8k_0^2}(B_{xx} - 2B_{yy}) + \frac{i}{8}B|B|^2
\]
\[
A_2 = -\frac{1}{2}B^2 + \frac{i}{k_0}BB_x
\]
\[
A_3 = -\frac{3i}{8}B^3
\]
Extension to nonlinear waves

- **Key result**: for moderate steepness, full wave height distribution again reasonably approximated by K-distribution with $N$ degrees of freedom.

- Wave height probability depends on single parameter $N$ (Rayleigh as $N \to \infty$).

- How does $N$ depend on wave steepness, incoming angular spread, spectral width, etc?
Steepness \( \varepsilon = k \cdot (\text{mean crest height}) \)

\[ \Delta \theta = 2.6^\circ \]

\[ \Delta k / k = 0.1 \]

\[ u = 0 \]
N as function of incoming angular spread for fixed steepness and frequency spread
N as function of incoming frequency spread for fixed steepness and angular spread
N as function of steepness for fixed incoming angular spread and frequency spread
Finally, combine currents and nonlinearity!
Summary

- Linear refraction of stochastic Gaussian sea produces lumpy energy density
  - Skews formerly Rayleigh distribution of wave heights
- Importance of refraction quantified by freak index $\gamma$
  - Spectacular effects in tail even for small $\gamma$
- Very similar wave height distribution in the presence of moderate nonlinearity
- Even more dramatic results obtained when random currents and nonlinearity are acting in concert
  - Refraction may serve as trigger for full non-linear evolution
Thank you!

Ying, Zhuang, Heller, and Kaplan,
Nonlinearity 24, R67 (2011)