Quantum Chaos: Electron Waves in Nanostructures and Freak Waves in the Ocean

Lev Kaplan
Tulane University
Talk outline:

- Chaos: what is it and why should we care?
  - Classical (ray) chaos
  - Quantum (wave) chaos
- Two applications: motion in random potential
  - Electron flow in nanostructures (10^{-7} m)
  - Freak waves on the ocean (10^5 m)
Classical (ray) chaos

- Newton’s laws: given $x(0)$, $v(0)$ & forces, in principle can calculate $x(t)$, $v(t)$ at all future times

- Regular system: small change in $x(0)$ or $v(0)$ will lead to small change in $x(t)$, $v(t)$ [at most, effect will grow linearly with $t$]
  - one-dimensional motion
  - two- or three-dimensional separable motion
    - projectile, particle in rectangular or circular box
    - spherical or cylindrical symmetry, …

- Fully regular behavior is unusual in $d \geq 2$
Classical (ray) chaos

- Instead, small change in initial conditions generally results in exponentially growing change with time: $\Delta x(t) \sim e^{\alpha t} \Delta x(0)$

- Suppose error doubles after each bounce ($\alpha = \ln 2$)
  - Require $\Delta x(t) < 1 \text{ m}$
  - For 10-bounce calculation, need $\Delta x(0) < 10^{-3} \text{ m}$ accuracy
  - For 50-bounce calculation, need $\Delta x(0) < 10^{-15} \text{ m}$ accuracy
  - For 150-bounce calculation, need $\Delta x(0) < 10^{-45} \text{ m}$ accuracy

- Unpredictable determinism! (no information produced)

- Example: weather forecasting (Lorenz)

- Need to focus on statistical properties, instead of computing individual trajectories
Quantum (wave) chaos

- Replace particle (or ray) bouncing in a box with analogous wave system: vibrating drumhead
- Wave system *not* chaotic
  - Exponential sensitivity to infinitesimal change in initial conditions washed out by finite wavelength (uncertainty principle)

\[ \int \psi^* (x,0) \psi (x,0) dx = 0.9 \Rightarrow \int \psi^* (x,t) \psi (x,t) dx = 0.9 \]

- But, correspondence principle (short wavelengths)
Quantum (wave) chaos

- Study of quantum (or classical wave) systems whose classical (or ray) limit is chaotic
- Look at spectral, wave function, transport properties
- In general, no analytic solutions
- One approach: brute-force numerical calculation
  - Little insight, need to re-do for new parameters
- Instead, we can
  - Search for general predictions of statistical nature, applicable to all quantum chaotic systems
  - Look for specific correspondence between properties of classical and quantum systems
  - Go beyond simplest “random wave” model, which predicts Gaussian random wave function fluctuations
Quantum (wave) chaos

- Key tool: semi-classical evolution
  - Quantum sum over all paths (Feynman) approximated by sum over classical paths with phases
  - Includes interference (double-slit) but not “hard quantum” effects such as tunneling, diffraction
  - Bridge between quantum mechanics and our classical intuition

- Bring together insights, methods, examples from AMO, nuclear, nanostructures, microwaves, acoustics, mathematical physics, …
Quantum (wave) chaos: examples

Quantum:
- Conductance through nanodevices
- Hydrogen in strong magnetic field
- Highly excited molecules
- Quantum corrals

Classical waves:
- Microwave resonators
- Acoustics: long-range sound propagation in ocean
- Optics: directed emission from microlasers
Conductance through tunneling diode
(Monteiro et al, Nature 1997)
Hydrogen atom wave function in strong B field
Wave functions for highly excited $\text{H}_2$ molecule
“Quantum corral”: STM image of surface electron density (Crommie et al, Science 1993)
Directional emission from microlaser with dielectric resonator (Gmachl et al, Science 1998)
Numerical calculations for stadium billiard
Application: Electron Flow in Nanostructures

- Electrons confined to 2-dimensional electron gas (2DEG) inside GaAs-AlGaAs heterostructure
- Gate voltages used to create barriers inside 2DEG and carve out region through which electrons may flow
Application: Electron Flow in Nanostructures

- Barriers chosen so electrons must pass through narrow quantum point contact (QPC)
- Scanning probe microscope technique used to image electron flow through such a 2DEG device
  - Negatively charged tip reflects current and reduces conductance through device
  - By measuring reduction in conductance as function of tip position, can map out regions of high and low current
Application: Electron Flow in Nanostructures

- Known: potential $V(x,y)$ away from gates is not zero, but is random due to donor ions and impurities in 3d bulk.

- Expected: outgoing flow should exhibit random wave pattern (Gaussian random amplitude fluctuations).

- Instead: observe strong current concentrated in small number of “branches” (Topinka et al, Nature 2001).
Understanding branching of electron flow

- Simplest description of random potential: Gaussian random with rms height $V_{\text{RMS}}$ and correlation distance $d$
- Numerical simulations of the Schrödinger equation show qualitatively similar branching behavior
- Also seen in classical simulation!
  - Semiclassical or ray picture must be applicable
  - Branches do not correspond to “valleys” of the potential
Understanding branching of electron flow

- Consider parallel incoming paths encountering a single shallow dip in potential $V(x,y)$
- Focusing when all paths in a given neighborhood coalesce at a single point (caustic), producing infinite ray density
- Different groups of paths coalesce at different points ("bad lens" analogy)
Understanding branching of electron flow

- Generic result: “cusp” singularity followed by two lines of “fold” caustics
- At each y after cusp singularity, we have infinite density at some values of x
Understanding branching of electron flow

- Realistic situation: weak potential $V_{\text{RMS}} \ll \text{KE} \Rightarrow$ small angle scattering

- Single bump or dip of size $\sim d$ insufficient to produce cusp singularity

- Instead, first singularities formed after typical distance scale $D \sim d \left(\text{KE}/V_{\text{RMS}}\right)^{2/3} \gg d$

- Further evolution: exponential proliferation of caustics
  - Tendrils decorate original branches
  - Universal branch statistics with single distance scale $D$

- Individual branch locations & heights depend on fine details of random potential, but statistics depend only on dimensionless parameter $\text{KE}/V_{\text{RMS}}$
Understanding branching of electron flow

- Multiple branching
More artistic visualization of electron flow

(Eric Heller)
Computing statistics of branched flow

- Wave mechanics: caustic singularity washed out on wavelength scale (uncertainty principle)
  - Visible branch only if smeared intensity above background
- At long distances $y \gg D$ from QPC, 2 effects
  - Number of caustics grows exponentially
  - Typical intensity of each caustic decays exponentially due to stretching of phase space manifold
- But not all pieces of manifold stretch at same rate
  - Visible branch occurs only when singularity occurs in piece of manifold that had stretched anomalously little
Computing statistics of branched flow

- When all distances expressed in terms of D, everything governed by 2 dimensionless numbers (describing log-normal distribution of stretching factors)
  - $\alpha$: average rate of stretching
  - $\beta$: variance of stretching rate
- Then # of branches decays exponentially as
  \[ \exp\left( - \frac{\alpha^2}{\beta} \frac{y}{D} \right) \]
- Intensity of strongest branch:
  \[ \ln (I_{\text{max}}) = (1/2) \ln \left( \frac{d}{\lambda} \right) - \gamma \left( \frac{y}{D} \right) \]
  where \( \gamma = \alpha - \frac{\beta}{2} \left[ \sqrt{1 + 4 \frac{\alpha}{\beta}} - 1 \right] \)
  and \( \lambda \) is the wavelength
Numerical simulations

- Intensity of strongest branch:

- Similarly can compute distribution of branch heights, fraction of space covered by branches, etc.
Application: Freak Waves
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- About 10 large ships lost per year to presumed rogue waves – usually no communication
- Also major risk for oil platforms in North Sea, etc.
- Probability seems to be much higher than what one would expect from Gaussian random model of wave heights
- Various possible explanations: nonlinear instabilities…
- Here, focus on refraction of incoming wave (velocity v) by random current eddies (typical current speed $u_{RMS} \ll v$)
Application: Freak Waves

Rogue Giants at Sea

By WILLIAM J. BROAD
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The storm was nothing special. Its waves rocked the Norwegian Dawn just enough so that bartenders on the cruise ship turned to the usual palliative — free drinks.

Then, off the coast of Georgia, early on Saturday, April 16, 2005, a giant, seven-story wave appeared out of nowhere. It crashed into the bow, sent deck chairs flying, smashed windows, raced as high as the 10th deck, flooded 62 cabins, injured 4 passengers and sowed widespread fear and panic.

"The ship was like a cork in a bathtub," recalled Celestine McElhatton, a passenger who, along with 2,000 others, eventually made it back to Pier 88 on the Hudson River in Manhattan. Some vowed never to sail again.
Different dispersion relation:
- Electron waves: $E \sim p^2 \Rightarrow \omega \sim k^2 \Rightarrow v \sim k$
- Surface water waves: $\omega \sim k^{1/2} \Rightarrow v \sim k^{-1/2}$

Also, initial spread of wave directions important
- New parameter $\Delta \theta$
- Causes smearing of singularities, on scales much larger than a wavelength

Ray dynamics produces regions of high/low intensity
- Within each region, wave heights given by Gaussian model (Rayleigh distribution)
- Overall wave height probability distribution obtained by averaging over high and low intensity regions
Calculation: Freak Waves

Wave height intensity distribution: \( P(h) = \int P_\sigma(h) f(\sigma) d\sigma \)

Obtained by superposing locally Gaussian wave statistics on pattern of “hot/cold spots” caused by refraction

\( \sigma^2(\vec{r}) \) is position-dependent variance of the water elevation (high in focusing regions, low in defocusing regions)

Probability \( f(\sigma) \) can be computed if “freak index” \( \gamma = \Delta\theta (v/u_{RMS})^{2/3} \) is known

\( P_\sigma(h) = (h/\sigma^2) \exp(-h^2/2\sigma^2) \) Rayleigh distribution
Implications for Freak Wave Statistics

- Simulations (using Schrödinger equation): long-time average and regions of extreme events

\[ \gamma \approx 2 \]
Typical ray calculation for ocean waves

\[ \Delta \theta = 5^\circ \]

\[ \Delta \theta = 25^\circ \]
Implications for Freak Wave Statistics

- Modified distribution of wave heights

\[ \gamma = 3.4 \]

Dashed = Rayleigh
Dotted = Theory based on locally Gaussian fluctuations

Note: significant deviations from Rayleigh in extreme tail
Summary:

- Quantum chaos: study of “generic” quantum (or classical wave) systems (i.e. lacking symmetries that make problem trivial)

- Essential tools:
  - Semiclassical methods (ray-wave correspondence)
  - Statistical approaches

- Applications:
  - Statistics of branched electron flow in 2DEG
  - Probability of freak wave encounters on the ocean