Quantum Vacuum Energy in Graphs and Billiards

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  - Basic Idea
  - Mathematical Setup
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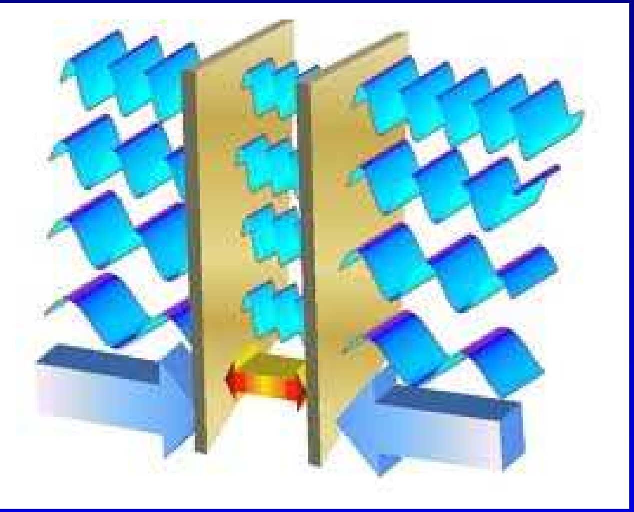
- Summary

History and Background

- Any quantum system has nonzero ground state energy, e.g. $\hbar\omega/2$ for harmonic oscillator with frequency ω
- In particular, vacuum state of a quantum field has nonzero energy, which is a function of boundary conditions
- 1948: Casimir-Polder force predicted between uncharged conducting plates

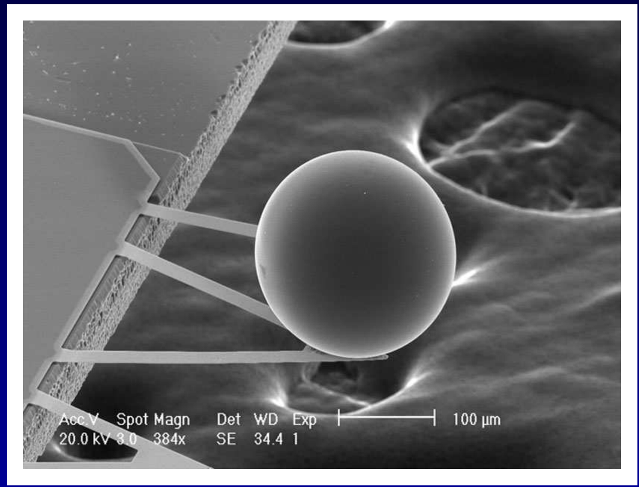
$$F/A = -\pi^2\hbar c/240a^4$$

Measurement in 1958 consistent with theory



History and Background

- 1997: More accurate experiment using plane and sphere



- 2002: Finally, original Casimir parallel plate experiment performed with 15% precision

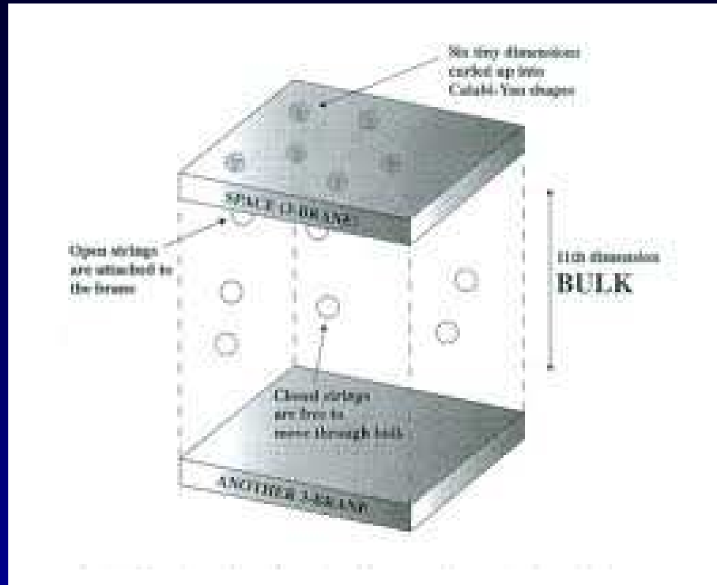
History and Background

Vacuum energy beyond parallel plates:

- Relation to van der Waals force in chemistry
- 1970s: QFT in curved spacetime
- Chiral bag model of the nucleon
- Dark energy in cosmology
  - $\Lambda \sim 10^{-26} kg/m^3 \sim 10^{-120} c^5/\hbar G^2$
  - Why so small?
  - Why not zero?
  - Why comparable to present-day mass density of universe?

History and Background

- Stabilizing extra dimensions in brane world models



- Nanotechnology applications: MEMS and NEMS

  Static friction caused by vacuum energy is major obstacle to further miniaturization of devices such as gears, etc.

Basic Idea

- Find all modes of field φ consistent with given boundary conditions, i.e. solve

$$-\nabla^2\varphi_n = \frac{\omega_n^2}{c^2}\varphi_n$$

- Each mode n behaves as independent harmonic oscillator with frequency ω_n
- Each mode has energy $\hbar\omega_n(N_n + \frac{1}{2}) \qquad N_n = 0, 1, 2, \ldots$
- Total vacuum energy of the field is

$$\frac{\hbar}{2}\sum_n \omega_n$$

Mathematical Setup

H = 2nd order, elliptic, self-adjoint operator (here $H = -\nabla^2$) acting on field φ in compact region $\Omega \subset R^n$

$H\varphi_n = \lambda_n \varphi_n$

Assume spectrum is nonnegative and discrete

Each mode n behaves as an independent harmonic oscillator with frequency $\omega_n = \sqrt{\lambda_n}$ $[c = 1]$

Zero-point energy of each mode is $\frac{1}{2}\omega_n$ $[\hbar = 1]$

Vacuum energy $E = \dfrac{1}{2}\sum_n \omega_n = \dfrac{1}{2}\sum_n \sqrt{\lambda_n} = \dfrac{1}{2}\,\mathrm{Tr}\,\sqrt{H}$

Mathematical Setup

$E = \frac{1}{2}\sum_n \omega_n = \frac{1}{2}\,\mathrm{Tr}\,\sqrt{H}$ divergent, must regularize

Cylinder (Poisson) kernel: $T_t(x,y) = \langle x|e^{-t\sqrt{H}}|y\rangle$

Let
$$E_t = -\frac{1}{2}\frac{\partial}{\partial t}\mathrm{Tr}\;T_t = \frac{1}{2}\sum_n \omega_n e^{-\omega_n t}$$

Somehow must take $t \to 0$ and get $t-$independent finite answer for physical forces [renormalization]

Similarly obtain energy density
$E_t(x, \xi = \frac{1}{4}) = -\frac{1}{2}\frac{\partial \mathrm{Tr}\;T_t}{\partial t}(x,x) = \frac{1}{2}\sum_n \omega_n |\varphi(x)|^2 e^{-\omega_n t}$
and other components of stress-energy tensor

Real life is more complicated

In specific applications need to consider

- vector fields (e.g., electromagnetic fields)
- going beyond idealized boundary conditions
- more physically realistic cutoffs (e.g., spatial dispersion)

Advantages of toy model

- can address general conceptual issues independent of specific application
- mathematical problem in spectral geometry: asymptotics of cylinder kernel, relation to zeta function, ...

Objectives/Questions

- Role of periodic and closed classical orbits
  - quantum-classical correspondence
- Sign of Casimir force in general situations
- Relation of local and global quantities
  - nonuniform convergence
- Systematic understanding of boundary, curvature, and corner effects

Objectives/Questions

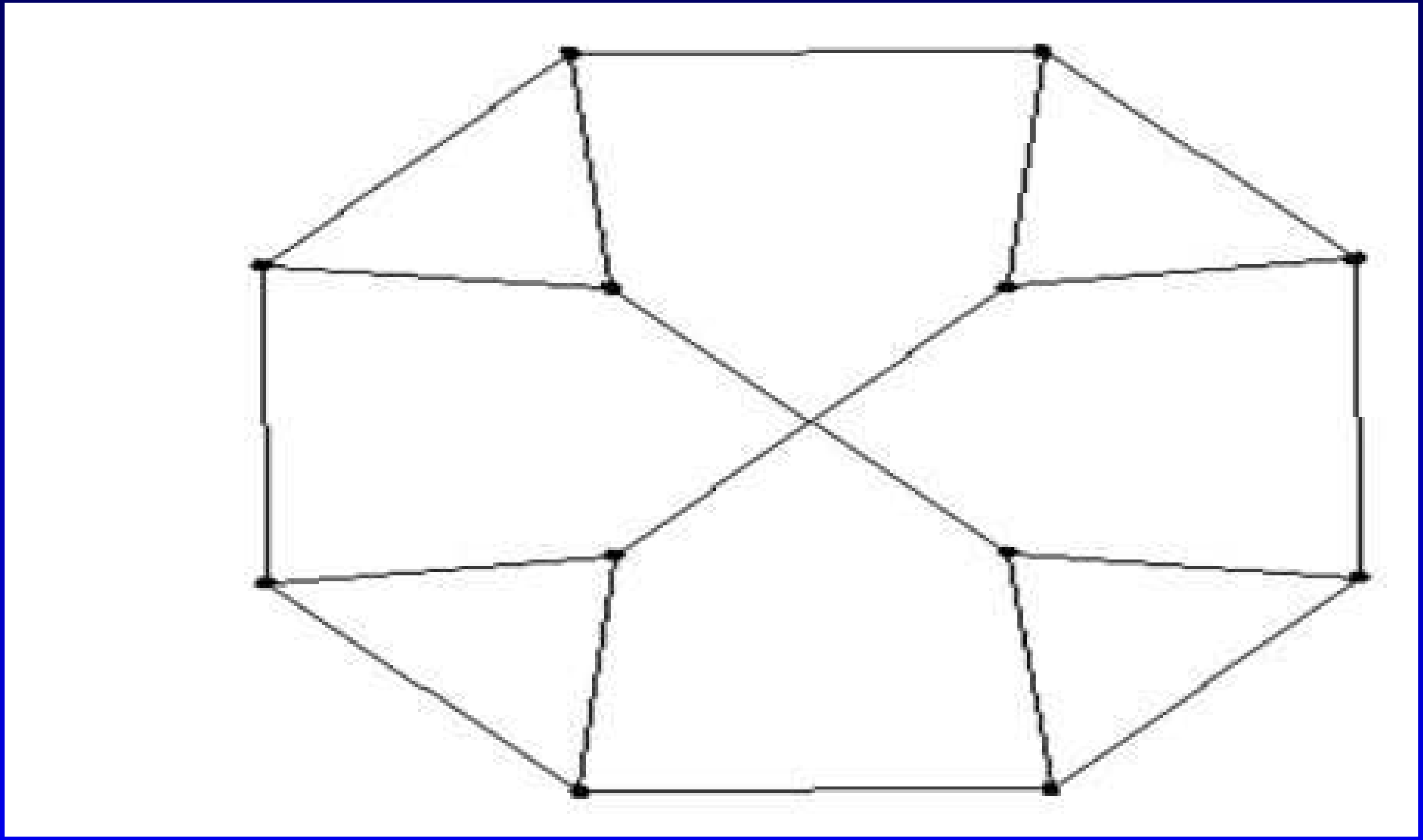
- Proper renormalization
  - under what circumstances do divergent terms cancel?
  - can all ∞'s be absorbed into boundary properties?
- Combining "inside" + "outside" contributions
- Coupling to gravity [*Estrada et al., J. Phys. A (2008)*]
- Cutoff theories and Lorentz invariance

Ex. 1: Quantum Graphs

S. A. Fulling, L.K., and J. H. Wilson, PRA (2007)

G. Berkolaiko, J. M. Harrison, and J. H. Wilson, J Phys A (2009)

J. H. Wilson, senior thesis



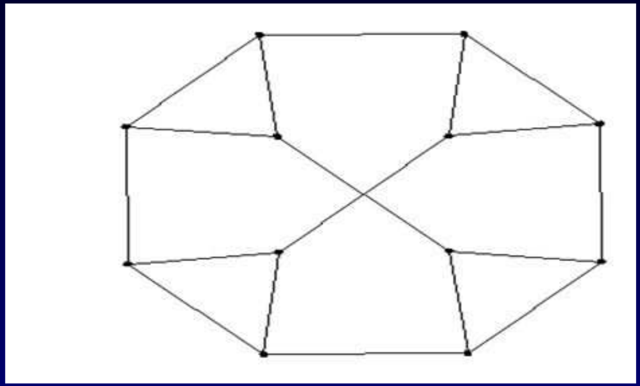
Ex. 1: Quantum Graphs

What is a quantum graph?

- Set of line segments joined at vertices
- Singular one-dimensional variety equipped with self-adjoint differential operator
- Approximation for realistic physical wave systems
  - Chemistry: free electron theory of conjugated molecules
  - Nanotechnology: quantum wire circuits
  - Optics: photonic crystals
- Laboratory for investigating general questions about scattering, quantum chaos, spectral theory

Ex. 1: Quantum Graphs

$H = -\nabla^2$ on each bond

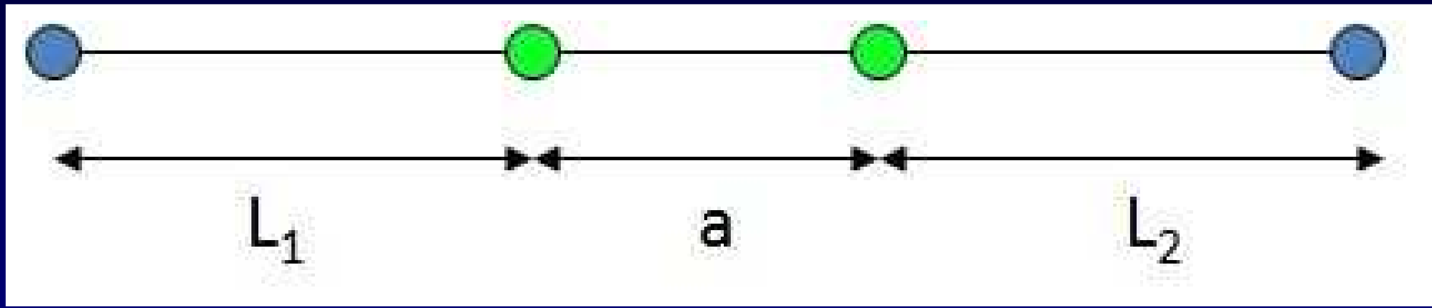


Kirchhoff boundary conditions at each vertex

- Continuity $\psi_j(0) = \psi_\alpha$ for all bonds j starting at vertex α
- Current conservation $\sum_j \partial\psi_j(0) = c_\alpha \psi_\alpha$ where sum is over all bonds j starting at vertex α, and derivative is in outward direction
- Neumann-like: $c_\alpha = 0$; Dirichlet: $c_\alpha = \infty$

Vacuum Energy in QG

Direct calculation using spectrum



Two movable "pistons"

Focus on "a" region between pistons:

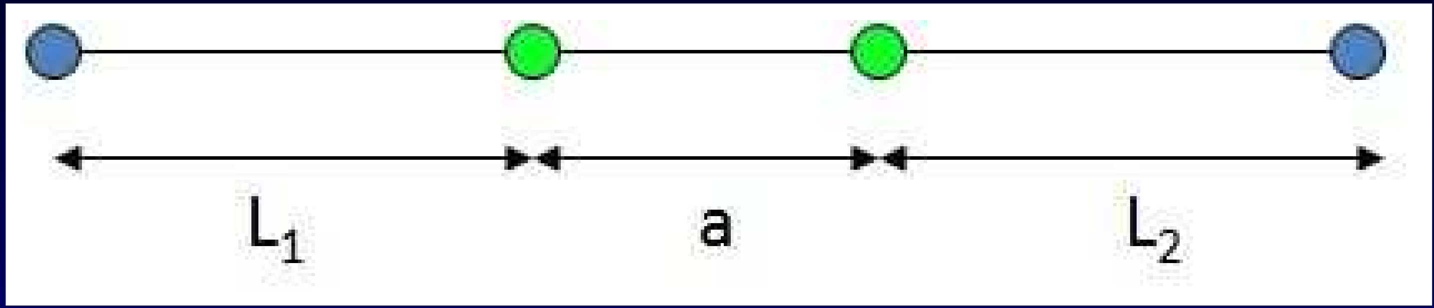
Dirichlet (DD): $\omega_n = n\pi/a \; (n = 1 \cdots \infty)$

Neumann (NN): $\omega_n = n\pi/a \; (n = 0 \cdots \infty)$

$$\mathrm{Tr}\; T_t = \sum_{n=0,1}^{\infty} e^{-(n\pi/a)t} = \frac{a}{\pi t} \pm \frac{1}{2} + \frac{1}{12}\frac{\pi t}{a} + O(t^2)$$

$$\Rightarrow E_t = -\frac{1}{2}\frac{\partial \mathrm{Tr}\; T_t}{\partial t} = \frac{a}{2\pi t^2} - \frac{\pi}{24a} + O(t)$$

Vacuum Energy in QG



$$E_t = \frac{a}{2\pi t^2} - \frac{\pi}{24a} + O(t)$$

First term divergent as $t \to 0$

$$E_t^{\text{Total}} = \frac{a+L_1+L_2}{2\pi t^2} - \frac{\pi}{24}(a^{-1} + {L_1}^{-1} + {L_2}^{-1}) + O(t)$$

$$= \frac{\text{const}}{2\pi t^2} - \frac{\pi}{24}(a^{-1} + {L_1}^{-1} + {L_2}^{-1}) + O(t)$$

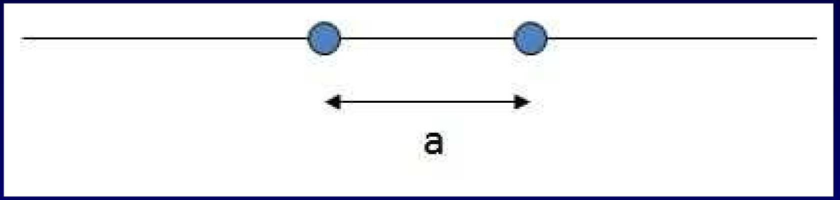
Divergent term is a-independent constant energy density

$\Rightarrow$ force on piston is finite!

Vacuum Energy in QG

$$E^{\text{Total}} = \text{const} - \tfrac{\pi}{24}(a^{-1} + {L_1}^{-1} + {L_2}^{-1})$$

Can safely take $L_1, L_2 \to \infty$



$$E_t^{\text{Total}} = \text{const} - \tfrac{\pi}{24a}$$

$$\Rightarrow F_{DD} = F_{NN} = -\frac{\partial E}{\partial a} = -\frac{\pi}{24a^2} \quad \text{(attractive)}$$

If *one* piston is Dirichlet and the other Neumann,

$$\Rightarrow F_{DN} = +\frac{\pi}{48a^2} \quad \text{(repulsive)}$$

Alternative Approach

Periodic Orbit Perspective: $\mathrm{Tr}\, T_t = \int dx\, T_t(x,x)$

- Free cylinder kernel in 1D: $T_t^0(x,y) = \frac{t}{\pi}\frac{1}{(x-y)^2+t^2}$
- Then $T_t(x,x)$ in problem with boundaries obtainable by method of images as sum over periodic and closed orbits:

$$T_t(x,x) = \mathrm{Re}\sum_p \frac{t}{\pi}\frac{A_p}{L_p^2+t^2} + \text{closed orbits} + O(t^2)$$

  - $p =$ periodic orbit going through x
  - $L_p =$ orbit length
  - $A_p =$ product of scattering factors

- Can obtain asymptotic $t \to 0$ behavior term by term

Alternative Approach

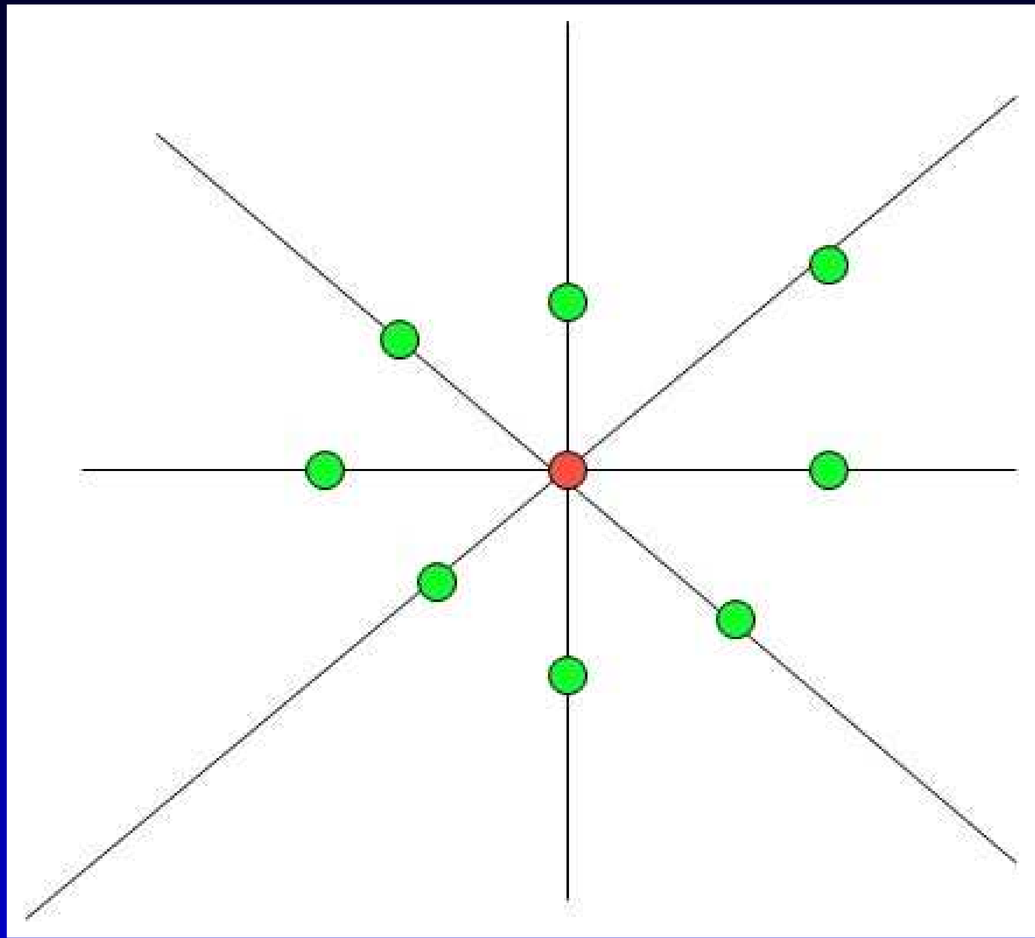
Periodic Orbit Perspective:

- Taking trace and accounting for repetitions r,
  $\mathrm{Tr}\, T_t = \int dx\, T_t(x,x) =$
  $\frac{t}{\pi}\frac{a}{t^2} + \mathrm{Re}\sum_p \sum_{r=1}^{\infty} \frac{t}{\pi}\frac{2L_p(A_p)^r}{(rL_p)^2} + O(t^2)$
- Divergent (Weyl) term associated with zero-length orbit
- For single line segment a, only one nonzero-length periodic orbit $L_p = 2a$ plus repetitions

$$F_{DD} = F_{NN} = -\frac{1}{4\pi a^2}\left(+1 + \frac{1}{4} + \frac{1}{9} + \cdots\right)$$
$$F_{DN} = -\frac{1}{4\pi a^2}\left(-1 + \frac{1}{4} - \frac{1}{9} + \cdots\right)$$

- Periodic orbit sum converges
- Sign of force can be read off from phase associated with shortest orbit (in general, boundaries + Maslov indices)

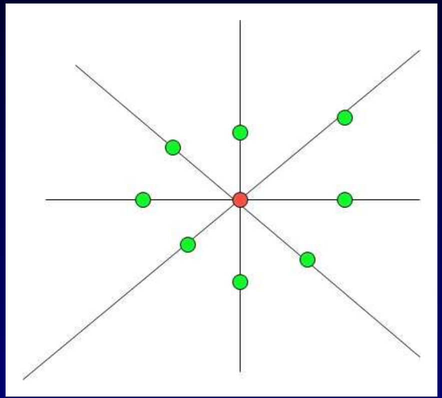
Vacuum Energy in Star Graphs



Vacuum Energy in Star Graphs

- N-like boundary at junction joining B bonds

- N, D, or $e^{i\theta}$ boundary at each piston



Each piston at distance a_j from junction

Exact expression for ω_n only when all a_j equal

In general, can find ω_n by solving characteristic equation $\det h(\omega) = 0$ numerically

$$\Rightarrow \quad \text{Then} \quad E = \lim_{t\to 0}\left[\frac{1}{2}\sum_n \omega_n e^{-\omega_n t} - \frac{\sum_j a_j}{2\pi t^2}\right]$$

Convergence improved by Richardson extrapolation

Star Graphs: Periodic Orbits

Alternatively: use periodic orbit expansion

Contribution to vacuum energy from shortest orbit only (bouncing back and forth once in one bond):

$$E \approx -\frac{1}{2\pi}\left(\frac{2}{B}-1\right)\sum_{j=1}^{B}\frac{(\pm 1)}{a_j}$$

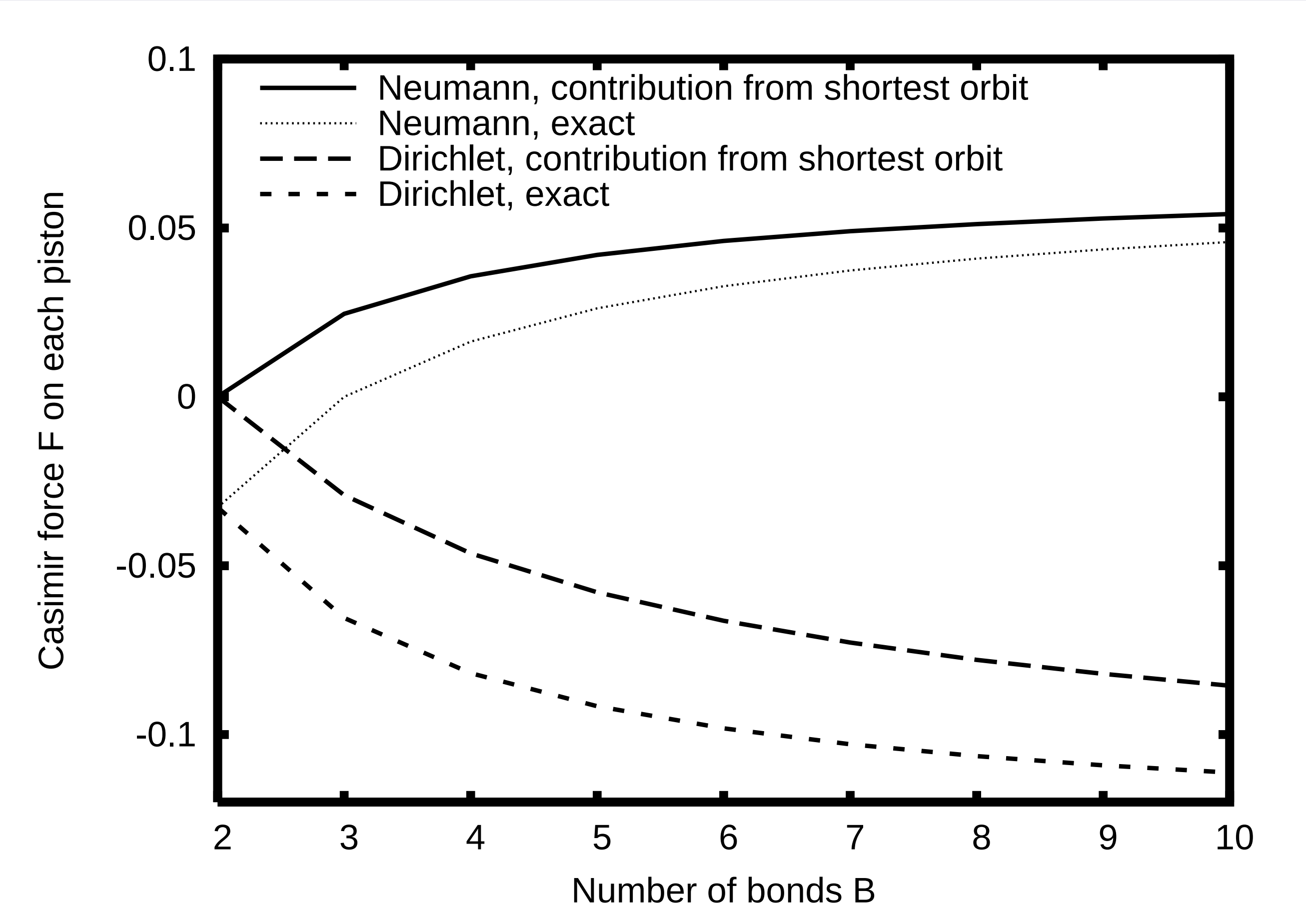
± 1 for Neumann or Dirichlet pistons

Gives correct sign for Casimir forces at least for $B > 3$

- repulsive for Neumann
- attractive for Dirichlet

Star Graphs: Periodic Orbits

Comparison between shortest orbit approximation & exact answer for equal bond case



Star Graphs: Periodic Orbits

Add up all repetitions of shortest orbits (Neumann):

$$E \approx -\frac{1}{4\pi} \sum_{r=1}^{\infty} \frac{1}{r^2} \left(\frac{2}{B} - 1 \right)^r \sum_{j=1}^{B} \frac{1}{a_j}$$

$$E \approx \frac{\pi}{48} \left(1 - \frac{24 \ln 2}{\pi^2 B} + \cdots \right) \sum_{j=1}^{B} \frac{1}{a_j}$$

Compare with analytic result for B Neumann pistons with *equal* bond lengths:

$$E = \frac{\pi}{48} \left(1 - \frac{3}{B} \right) \frac{B}{a}$$

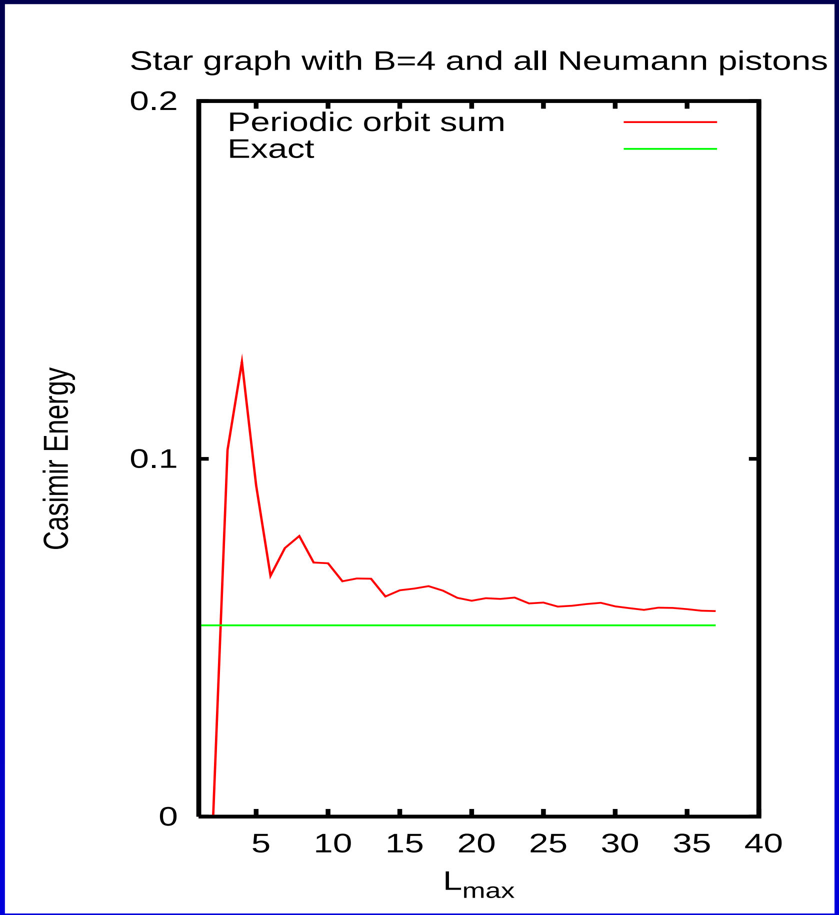
Shortest orbits give only leading contribution in $1/B$ expansion

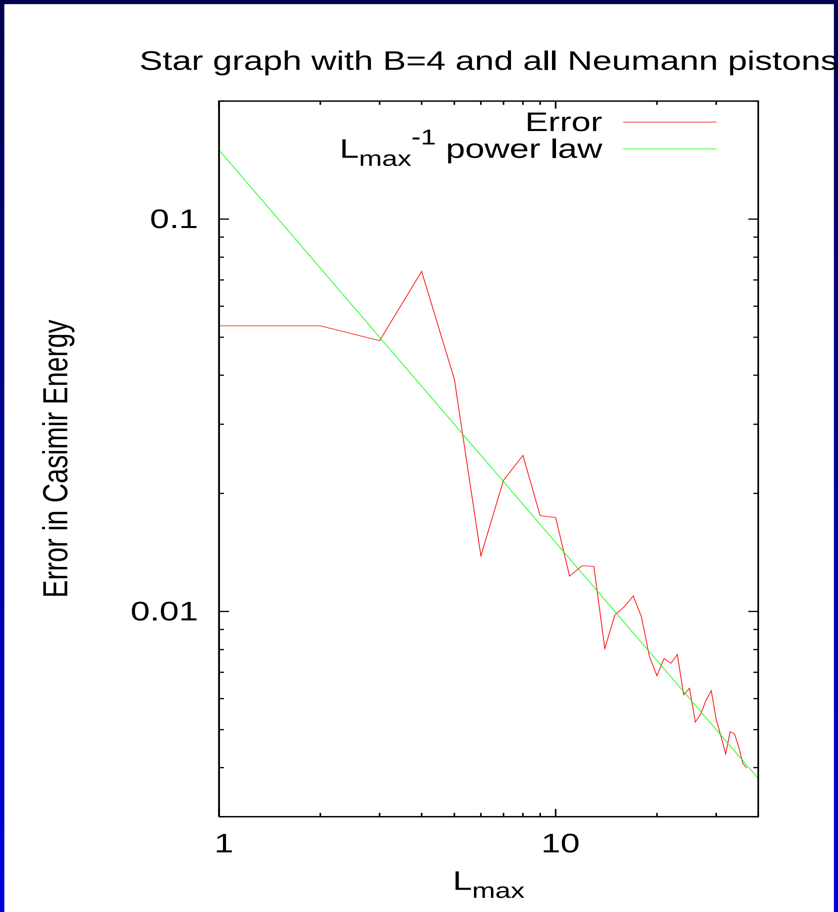
$\Rightarrow$ need more orbits to obtain full answer for finite B

Star Graphs: General Case

$B = 4$ star graph with unequal bonds and all Neumann pistons

Sum over orbits $L_p \leq L_{\max}$ & compare with “exact” answer



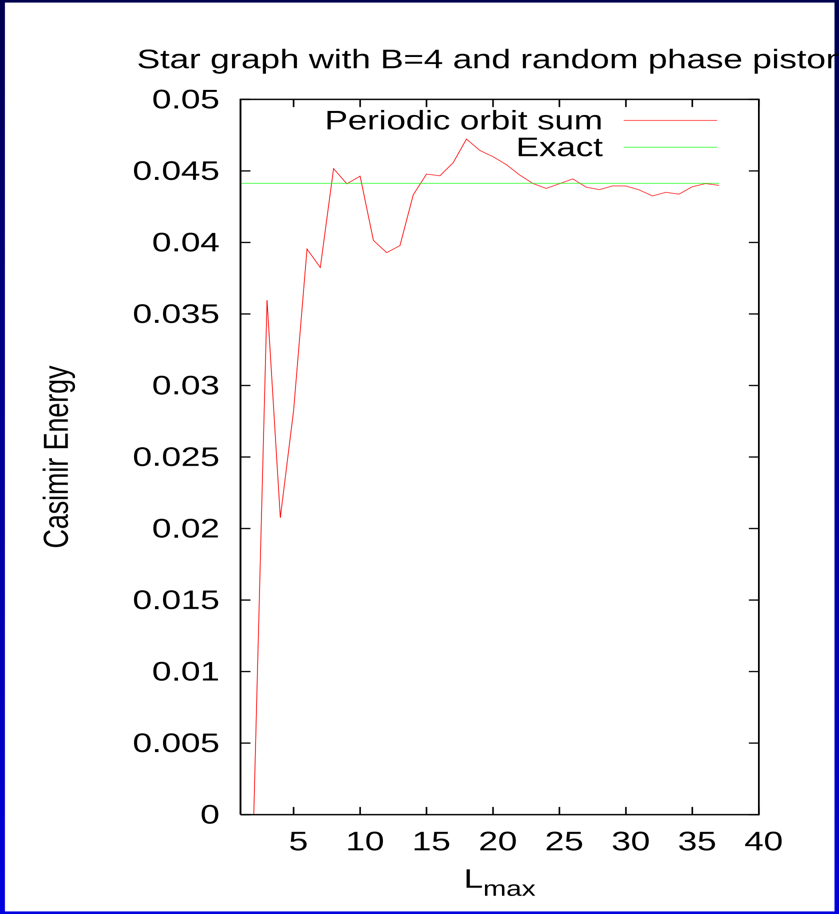


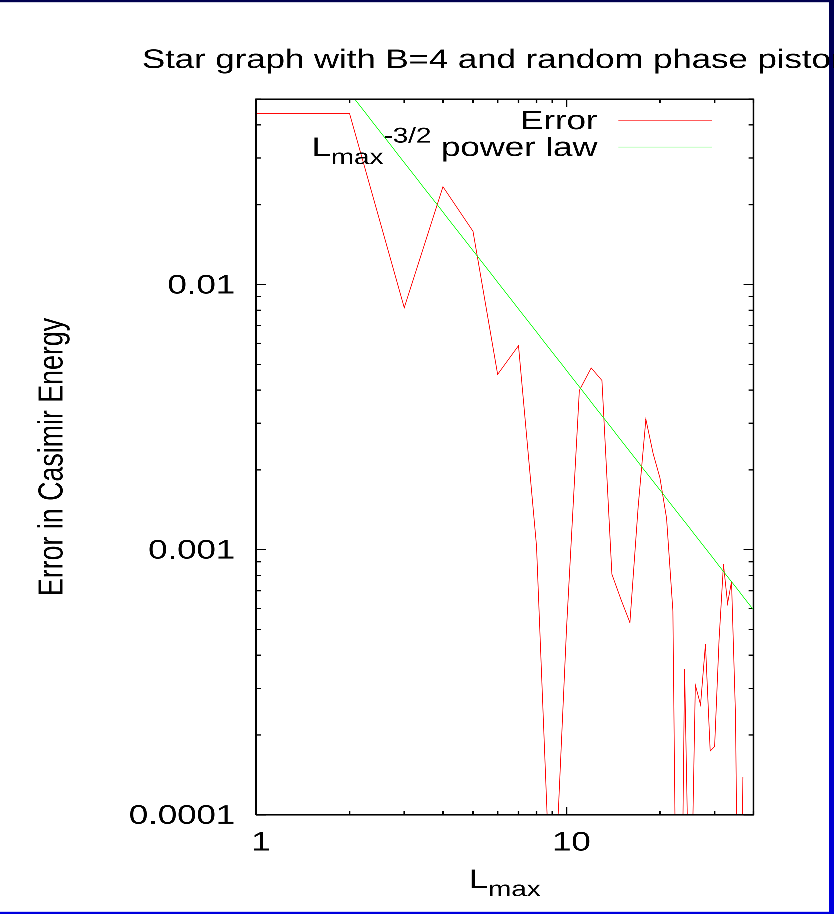
$$\text{Error} \sim (L_{\max})^{-1}$$

Star Graphs: General Case

$B = 4$ star graph with unequal bonds and arbitrary $e^{i\theta}$ pistons

Sum over orbits $L_p \leq L_{\max}$ & compare with “exact” answer

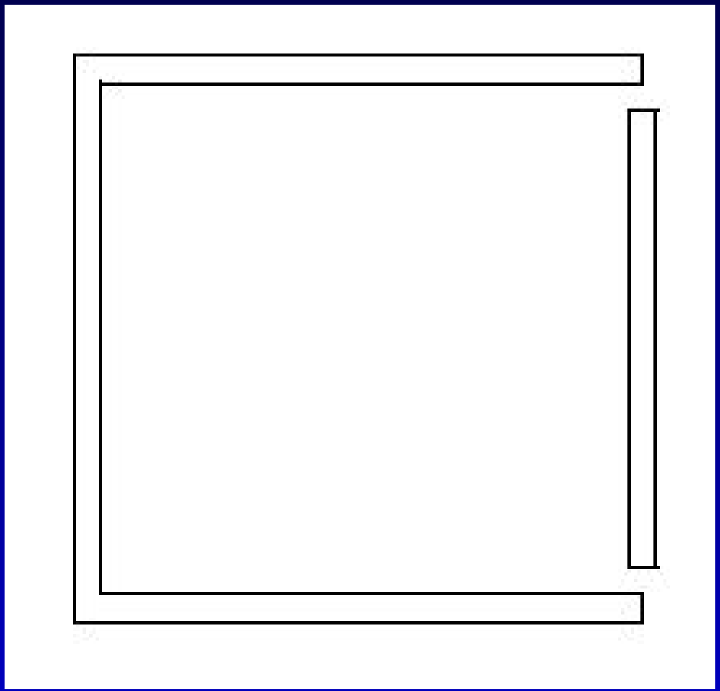




$$\text{Error} \sim (L_{\max})^{-3/2}$$

Ex. 2: Rectangles, Pistons, and Pistols

S. A. Fulling, L.K., K. Kirsten, Z. H. Liu, and K. A. Milton, J Phys A (2009);   Z. H. Liu, Ph.D. thesis



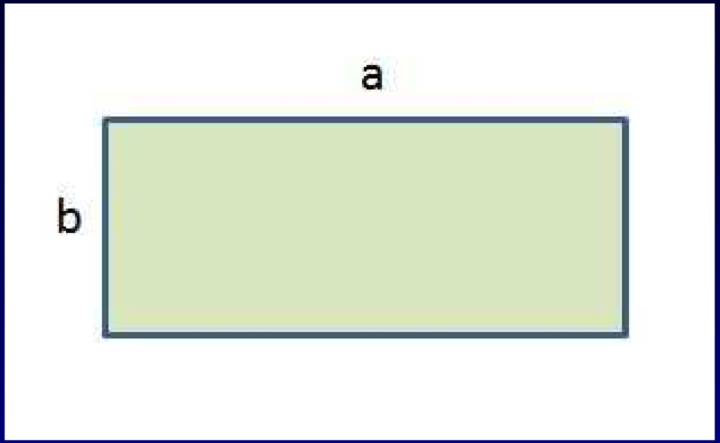
Motivating Question: Naive renormalization (Lukosz 1971, ...) suggests outward force on sides of square or cubic box

Is this force real?

Ex. 2a: Rectangular cavity

No straightforward way to evaluate $E_t = \frac{1}{2}\sum_n \omega_n e^{-\omega_n t}$ directly

Use classical path approach



Need all classical paths from x to x

Classify by number of bounces from a sides and number of bounces from b sides

- Periodic paths: Even Even
- Side paths: Even Odd or Odd Even
- Corner paths: Odd Odd

Rectangle: periodic paths

$$
\begin{aligned}
E_{t,\mathrm{Periodic}} &= \frac{ab}{2\pi t^3} - \frac{ab}{2\pi}\sum_{k=1}^{\infty}(-1)^{\eta}\frac{(2kb)^2-2t^2}{[t^2+(2kb)^2]^{5/2}} \\
&\quad -\frac{ab}{2\pi}\sum_{j=1}^{\infty}(-1)^{\eta}\frac{(2ja)^2-2t^2}{[t^2+(2ja)^2]^{5/2}} \\
&\quad -\frac{ab}{\pi}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{\eta}\frac{(2ja)^2+(2kb)^2-2t^2}{[t^2+(2ja)^2+(2kb)^2]^{5/2}}
\end{aligned}
$$

$\eta = \#$ of Dirichlet bounces

Assume all Neumann or all Dirichlet sides:

$$
\begin{aligned}
E_{t,\mathrm{Periodic}} &= \frac{ab}{2\pi t^3} - \frac{\zeta(3)}{16\pi}\left(\frac{a}{b^2}+\frac{b}{a^2}\right) \\
&- \quad \frac{ab}{8\pi}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(a^2j^2+b^2k^2\right)^{-3/2} + O(t^2)
\end{aligned}
$$

Rectangle: add all paths

$$E_{t,\mathrm{Nonperiodic}} \;=\; \mp\frac{2a+2b}{8\pi t^2} \pm \frac{\pi}{48}\left(\frac{1}{a}+\frac{1}{b}\right) + O(t^2)$$

Combining all terms:

$$\begin{aligned} E_t \;=\; & \frac{\text{Area}}{2\pi t^3} \mp \frac{\text{Perimeter}}{8\pi t^2} - \frac{\zeta(3)}{16\pi}\left(\frac{a}{b^2}+\frac{b}{a^2}\right) \\ - \; & \frac{ab}{8\pi}\sum_{j,k=1}^{\infty}\left(a^2j^2+b^2k^2\right)^{-3/2} \pm \frac{\pi}{48}\left(\frac{1}{a}+\frac{1}{b}\right) + O(t^2) \end{aligned}$$

Force on horizontal side:

$$\begin{aligned} F = -\frac{\partial E_t}{\partial a} = \text{divergent} + \frac{\zeta(3)}{16\pi b^2} - \frac{\zeta(3)b}{8\pi a^3} \\ + \frac{b}{8\pi}\sum_{j,k=1}^{\infty}\frac{k^2b^2-2j^2a^2}{(j^2a^2+k^2b^2)^{5/2}} \pm \frac{\pi}{48a^2} \end{aligned}$$

Force on side of rectangle

$$F = \text{divergent} + \frac{\zeta(3)}{16\pi b^2} - \frac{\zeta(3)b}{8\pi a^3}$$
$$+\frac{b}{8\pi}\sum_{j,k=1}^{\infty}\frac{k^2b^2 - 2j^2a^2}{(j^2a^2 + k^2b^2)^{5/2}} \pm \frac{\pi}{48a^2}$$

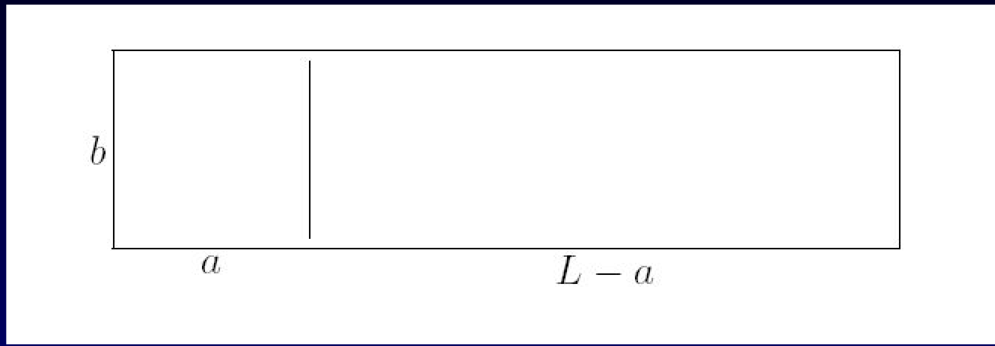
Naive “renormalization”: drop divergent terms and interpret $t-$independent result as physical force on the side of box

- $a << b$: attractive (like parallel plates)
- Square box with Dirichlet sides: repulsive!

Problems:

- Throwing away infinite terms
- Ignoring outside of box

Ex. 2b: Rectangular piston

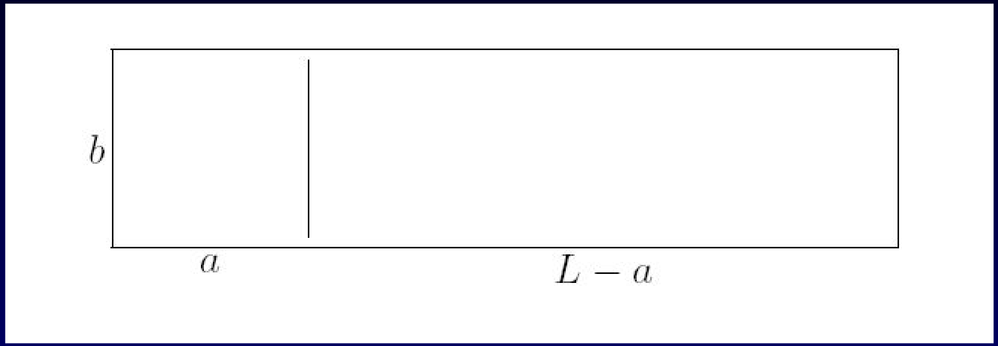


Add contributions from $a \times b$ rectangle and $(L-a) \times b$ rectangle

- Divergent terms cancel (total area and perimeter are conserved)
- One finite contribution from $(L-a) \times b$ rectangle survives as $L \to \infty$

$$F_{\text{piston}} = 0 + \frac{\zeta(3)}{16\pi b^2} - \frac{\zeta(3) b}{8\pi a^3} + \frac{b}{8\pi} \sum_{j,k=1}^{\infty} \frac{k^2 b^2 - 2 j^2 a^2}{(j^2 a^2 + k^2 b^2)^{5/2}} \pm \frac{\pi}{48 a^2} - \frac{\zeta(3)}{16\pi b^2}$$

Ex. 2b: Rectangular piston



Finally (Cavalcanti 2004)

$$F_{\text{piston}} = -\frac{\zeta(3)b}{8\pi a^3} + \frac{b}{8\pi}\sum_{j,k=1}^{\infty}\frac{k^2b^2 - 2j^2a^2}{(j^2a^2 + k^2b^2)^{5/2}} \pm \frac{\pi}{48a^2}$$

$$= \frac{\pi}{b^2}\sum_{j,k=1}^{\infty} k^2 K_1'\left(2\pi jk\,\frac{a}{b}\right) \qquad \text{[always attractive!]}$$

$$= -\frac{\zeta(3)b}{8\pi a^3} + \frac{\pi}{48a^2} - \frac{\zeta(3)}{16\pi b^2} + \frac{\pi b}{a^3}\sum_{j,k=1}^{\infty} k^2 K_0\left(2\pi jk\,\frac{b}{a}\right)$$

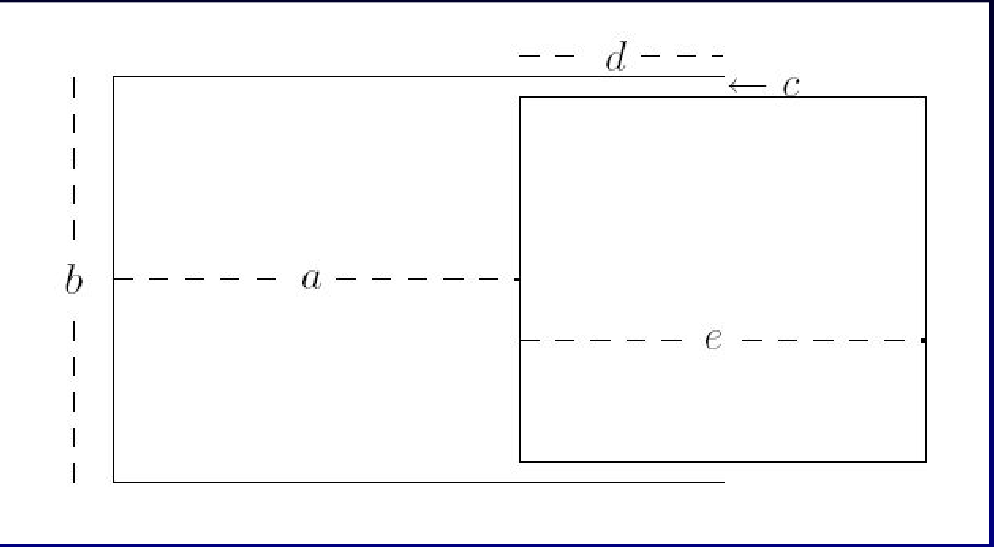
Decays exponentially for $a \gg b$; parallel plates for $a \ll b$

Ex. 2c: Casimir pistol

What would happen if external shaft is not present?

Here all dimensions $\gg$ cutoff t, except possibly $c \sim t$

All Dirichlet boundaries



$$E = \frac{us}{\pi t}\sum_{k=1}^{\infty}\frac{1-2k^2u^2}{(1+4k^2u^2)^{5/2}} + \frac{us}{\pi t}\sum_{j=1}^{\infty}\frac{1-2j^2s^2}{(1+4j^2s^2)^{5/2}}$$

$$+ \frac{2us}{\pi t}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1-2j^2s^2-2k^2u^2}{(1+4j^2s^2+4k^2u^2)^{5/2}}$$

$$+ \frac{s}{2\pi t}\sum_{j=1}^{\infty}\frac{-1+4j^2s^2}{(1+4j^2s^2)^2} + \frac{2r(l-s)}{\pi t}\sum_{k=1}^{\infty}\frac{1-2k^2r^2}{(1+4k^2r^2)^{5/2}}$$

Ex. 2c: Casimir pistol

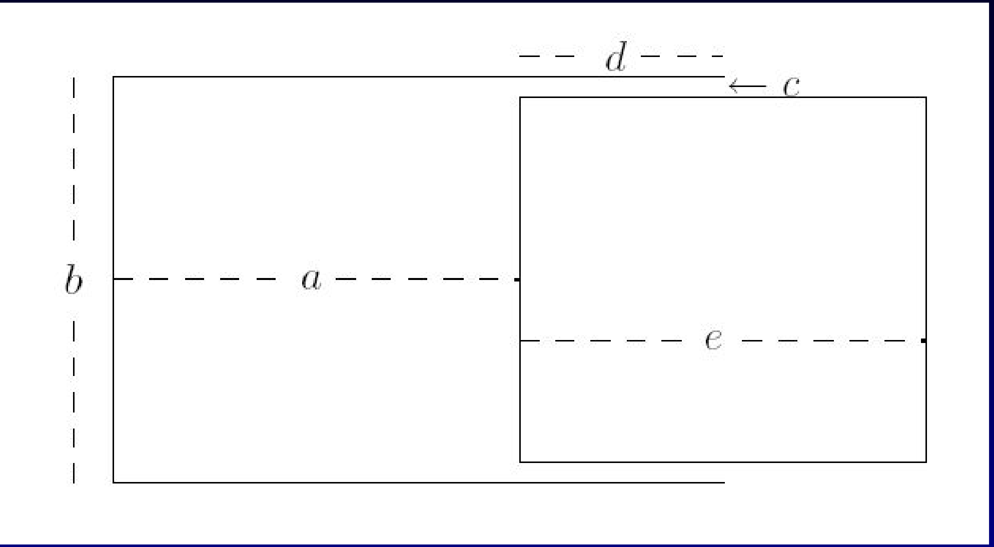
Use scaled variables

$c = rt$

$a = st$

$b = ut$

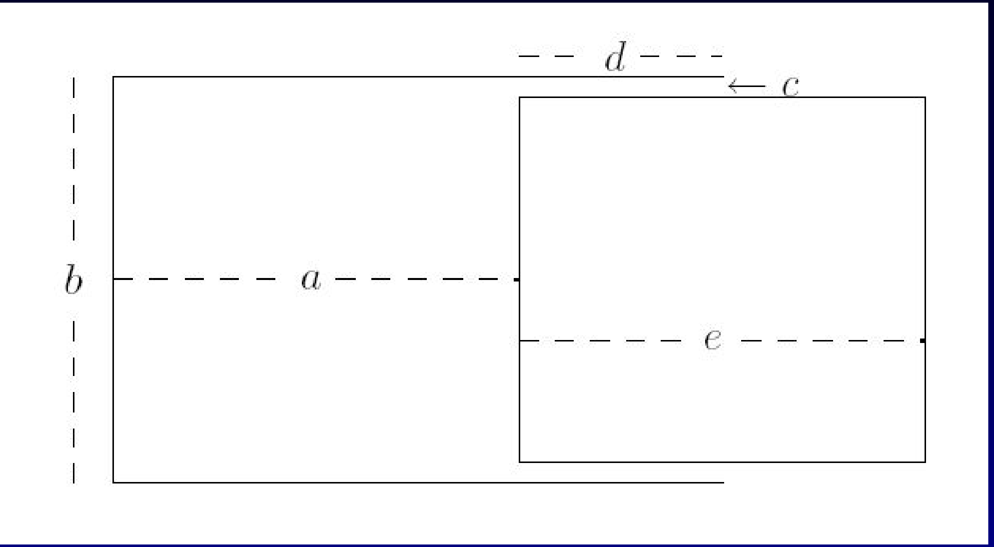
$d = L - a = (l - s)t$



$$
\begin{aligned}
E \;=\;& \frac{us}{\pi t}\sum_{k=1}^{\infty}\frac{1-2k^2u^2}{(1+4k^2u^2)^{5/2}} + \frac{us}{\pi t}\sum_{j=1}^{\infty}\frac{1-2j^2s^2}{(1+4j^2s^2)^{5/2}} \\
&+ \frac{2us}{\pi t}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\frac{1-2j^2s^2-2k^2u^2}{(1+4j^2s^2+4k^2u^2)^{5/2}} \\
&+ \frac{s}{2\pi t}\sum_{j=1}^{\infty}\frac{-1+4j^2s^2}{(1+4j^2s^2)^2} + \frac{2r(l-s)}{\pi t}\sum_{k=1}^{\infty}\frac{1-2k^2r^2}{(1+4k^2r^2)^{5/2}}
\end{aligned}
$$

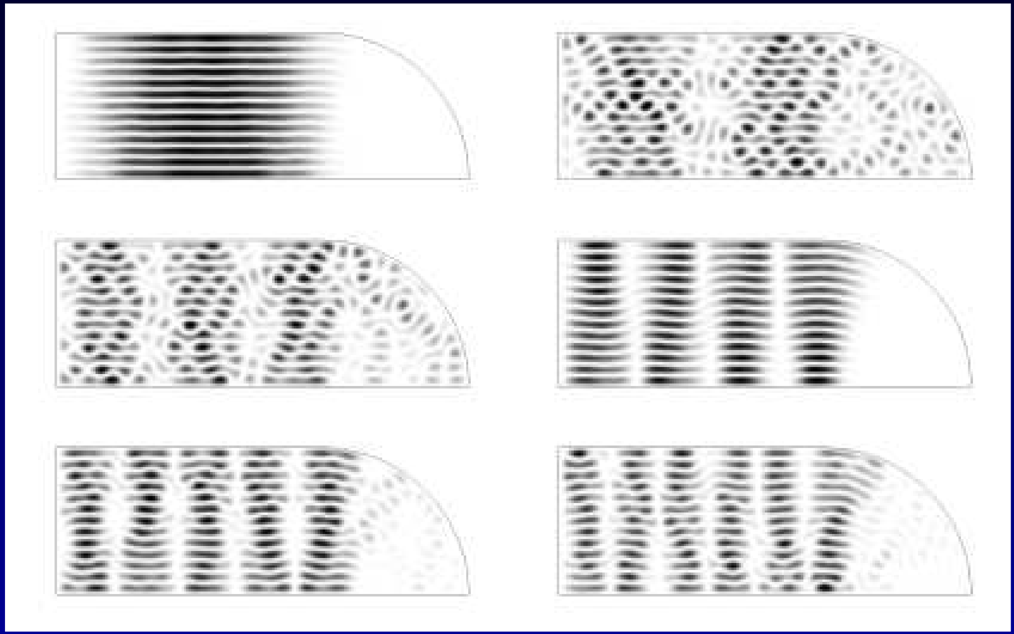
Ex. 2c: Casimir pistol

Horizontal force as function of a:



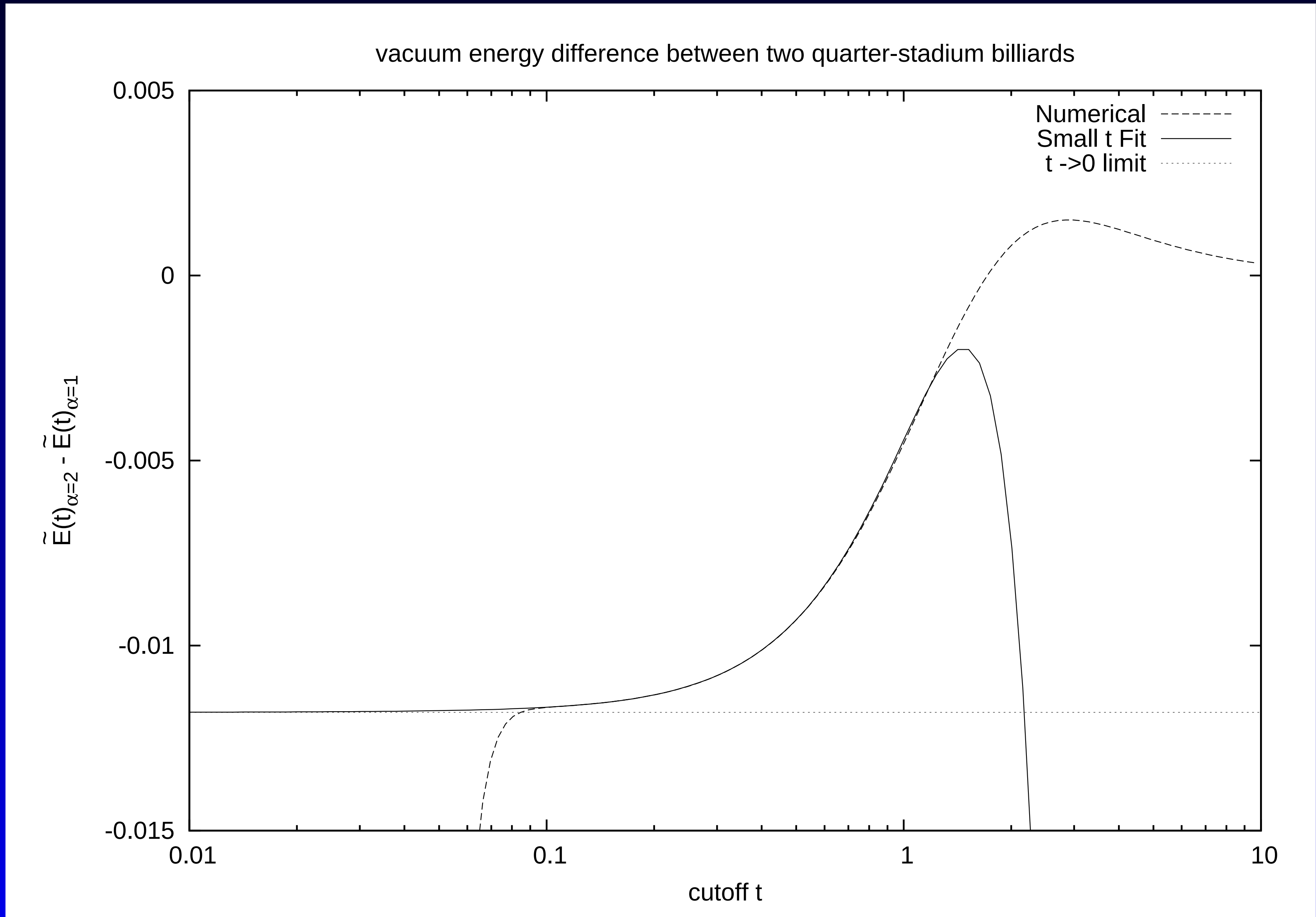
- narrow chamber $a \ll b^{1/3}c^{2/3}$: like parallel plates $F \sim -1/a^2$ (attractive)
- longer chamber $a \gg b^{1/3}c^{2/3}$: gaps dominate: F is $a-$independent
  - $c > 0.6t \Rightarrow$ attractive
  - $c < 0.6t \Rightarrow$ repulsive (believable???)

Ex. 3: Quarter stadium cavity



- Numerically obtain all frequencies ω_n up to $\omega_{\max}$
- Evaluate $E_t = \frac{1}{2}\sum_{\omega_n<\omega_{\max}} \omega_n e^{-\omega_n t} + O(e^{-\omega_{\max} t})$
- Leading $t \to 0$ behavior: $E_t^{\text{Weyl}} = \frac{\text{Area}}{2\pi t^3} - \frac{\text{Perimeter}}{8\pi t^2}$
- $E_t - E_t^{\text{Weyl}} = A \ln t + B + Ct + Dt^2 \ln t + Et^2 + Ft^3 + \cdots$
- Casimir force given by dependence of B on geometry

Ex. 3: Quarter stadium cavity



Ex. 4: Elliptic cavity

with H.-J. Flad and K. Kirsten

$$E_t = \frac{2a_0}{t} + \frac{a_{1/2}}{t^2} + E_0 + \frac{a_{3/2}}{2\sqrt{\pi}}(\gamma + \ln t) + \frac{a_2 t}{2} + \cdots$$

to be compared with heat kernel expansion

$$K(t) = \sum_n e^{-\omega_n^2 t} = \sum_{\ell=0,\frac{1}{2},1,\cdots} a_\ell t^{\ell-1}$$

with coefficients known analytically for this case in terms of hypergeometric functions

- Comparison with known heat kernel asymptotics allows check of numerics
- Then easily numerically obtain $E_0 = \frac{1}{2}\mathrm{FP}\zeta\left(-\frac{1}{2}\right)$

Conclusions

- Careful regularization and renormalization (including inside and outside contributions) needed to obtain physically meaningful energies and forces
- Classical orbit approach produces exact results in simple cases and may allow for good approximations where exact solutions are nonexistent
- Hope for intelligent combination of anlaytical and numerical tools for general geometries