Homework III: DENSITY DISTRIBUTION IN THE EARTH

1. Imagine the Earth to be a homogeneous, self-gravitating sphere of radius R and a uniform density ρ

   a. How does g vary inside this model of the earth as a function of the distance r from the center?

   b. What is the value of g at the center of this model of the earth?

   c. Derive an expression for pressure at the center of this model of the earth?

   d. Given that, $\bar{\rho} = 5{,}520\ kg/m^3$; $R_e = 6{,}371$ km; $G = 6.67 \times 10^{-11} m^3 kg^{-1} sec^{-2}$, calculate the pressure at the center of the earth if it were homogeneous.
   Express your answer in both Megabars and Gigapascals.
   (1 Pascal = 1 N $m^{-2} = 10^{-5}$ bars) (1N = 1 kgm/s^2) –see note on units at the end of this document.

   e. Why is the pressure from this model different from the actual pressure of approximately 3.6 Megabars?

2. An equation of state describes how physical properties vary as a function of conditions, for example pressure and temperature in the Earth's interior. Sometimes equations of state are derived from statistical mechanics, in which case the terms that appear in the equation relate to physical interactions that may take place as a result of changing conditions. Other equations of state are empirical. An empirical equation of state is simply a mathematical relationship that best fits experimentally determined data. The Murnaghan equation of state is one such empirical relationship that describes how the adiabatic bulk modulus changes with pressure for depths between 670 km and 1500 km in the upper mantle.

$$K = K_0 + K'_0 P$$

where $K_0 = 2.25 \times 10^{11}$ Pa and $K'_0 = 3.35$.

   a. using this equation of state and the relation

$$K = \rho dP/d\rho$$

   show that

$$\frac{\rho}{\rho_0} = \left[1 + \frac{K'_0 P}{K_0}\right]^{\frac{1}{K'_0}} = \left(K/K_0\right)^{\frac{1}{K'_0}}$$

where ρ_0 denotes the density of the upper mantle at zero pressure (uncompressed density.

b. Calculate ρ/ρ_0 at a pressure of 4.2×10^{10} Pa (0.42 Mbar).

c. Assuming the uncompressed density for this region (ρ_0) to be 4.1×10^3 kg m^{-3}, calculate the density (ρ) for this pressure.

3. The following table gives data for density, ρ, P-wave velocity, V_P, and S-wave velocity, V_S, at various depths in the Earth.

Depth (km)	ρ(kg/m^3)	V_P (km/s)	V_S (km/s)
120	3400	8.0	4.30
220	3400	7.8	3.80
2898	5,680	13.8	7.20
2902	9430	8.0	0.0
5198	14200	10.3	0.0
5202	16800	10.9	3.70

a. For each depth, calculate values for K, the bulk modulus, and μ, the shear modulus.

b. Give possible explanations for the data in the table and the values of K and μ that you calculated in part a.

c. Use Birch's Law to calculate the density at 120, 220, and 2898 km depth. $\rho = -1.87 + 0.43V_P$, where V_P is in units of m/s, and ρ has units of kg/m^3.

d. Compare the densities given in the table with those calculated using Birch's Law. How well do they agree? If they do not agree, can you suggest reasons why?

4. Consider a spherical body of radius r_a with a core of radius r_c and a constant density, ρ_c, surrounded by a mantle of constant density ρ_m.

a. using the relations

$$C = 8\pi/3\int \rho(r) r^4\, dr$$

and

$$M = 4\pi\int \rho(r) r^2\, dr$$

show that the principal moment of inertia C and the mass M are given by

$$C = 8\pi/15[\rho_c r_c^5 + \rho_m (r_a^5 - r_c^5)]$$

$$M = 4\pi/3[\rho_c r_c^3 + \rho_m (r_a^3 - r_c^3)]$$

b. From the expressions derived above, determine mean values for the densities of the Earth's mantle and core given

$C = 8.04 \times 10^{37}$ kg m^2, $M = 5.97 \times 10^{24}$ kg, $r_a = 6378$ km, and $r_c = 3486$ km.

c. Show that if the earth consisted of a mantle of density 4,500 kg/m^3 and thickness $r_a/2$ and a core of density 12,500 kg/m^3 of thickness $r_a/2$, that the total mass and average density would be equal to that of a homogeneous sphere of radius r_a and density 5,500 kg/m^3 throughout.

d. Calculate $C/(MR^2)$ for the equal thickness model in c above. Compare the calculated value with the observed value of 0.3308. What do you conclude about the relative merits of the homogeneous earth model versus the heterogeneous earth model?

5. The change in density with pressure under adiabatic conditions is given by

$$\beta_a = 1/\rho(d\rho/dP)_s$$

where s indicates that no heat is exchanged during the process.

In addition, the change in pressure with depth is given as

$$dP/dz = g\rho$$

a. assuming β_a is constant and $\rho = \rho_0$ at $P = 0$, integrate the first equation to get an expression of ρ as a function of P.

b. substitute the resulting expression into the second equation to find an expression for the pressure as a function of depth (assume g is constant and $P = 0$ at $z = 0$).

c. Given that $\beta_a = 4.3 \times 10^{-12}$/Pa, $g = 10$ m/sec^2, and $\rho_0 = 3310$ kg/m^3, calculate ρ and P in the mantle at a depth of 500 km.

Special Note on Units

For many years geologists have worked using the cgs system of units (i.e centimeters, grams, seconds). Within the last 10 years, however, the rest of the scientific community has switched to use the international system of units, SI (mks, meters, kilograms, seconds). For most geologists it easier to think in terms of cgs units, but at some point we must convert to SI. Given below are some of the conversion factors we must keep in mind to be consistent with SI units.

Density

Density in cgs units is given as g/cm^3 and SI units as kg/m^3.

$1\ g/cm^3 = 10^3 kg/m^3$

Pressure

Pressure is often given in units of bars or kilobars (kb). In cgs units,

$1\ bar = 10^6\ dynes/cm^2 \approx 1$ atmosphere

$1\ kb = 10^3$ bars, 1 Mb (megabar) $= 10^6$ bars

$1\ dyne = 1\ g\ cm/s^2$

In the SI units, pressure is given in Pascals, Pa

$1Pa = 1\ N/m^2$, where N refers to a Newton

$1N = 1kg\ m/s^2$.

To convert between the two systems we need also know that

$1Pa = 10\ dynes/cm^2 = 10^{-5} bars = 10^{-8}\ kb = 10^{-11}\ Mb$