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EENS 634 The Earth

Homework VII: Mantle Evolution and Heat Flow

1. Melting events in the mantle can cause fractionation of trace elements. For radiogenic isotopes this is important because it can change the ratios of parent to daughter isotopes such that their subsequent evolution will be different after the melting event. For example, partial melting of the mantle will result in a residual mantle with a higher Sm/Nd (thus higher $^{147}Sm/^{144}Nd$) ratio than the original mantle and the resulting melt having a lower Sm/Nd (thus lower $^{147}Sm/^{144}Nd$) than the original mantle. Similarly, a melting event will leave the mantle with a lower Rb/Sr (and thus lower $^{87}Rb/^{86}Sr$) ratio than the original mantle and the resulting melt will have a higher Rb/Sr (thus higher $^{87}Rb/^{86}Sr$) than the original mantle.

   a. Given the following equations and the data listed below

$$\left(\frac{^{143}\mathrm{Nd}}{^{144}\mathrm{Nd}}\right)_t = \left(\frac{^{143}\mathrm{Nd}}{^{144}\mathrm{Nd}}\right)_0 - \left(\frac{^{147}\mathrm{Sm}}{^{144}\mathrm{Nd}}\right)_0 \left(e^{\lambda_{147}t} - 1\right)$$

$$\left(\frac{^{87}\mathrm{Sr}}{^{86}\mathrm{Sr}}\right)_t = \left(\frac{^{87}\mathrm{Sr}}{^{86}\mathrm{Sr}}\right)_0 - \left(\frac{^{87}\mathrm{Rb}}{^{86}\mathrm{Sr}}\right)_0 \left(e^{\lambda_{87}t} - 1\right)$$

   where the t subscripts refer to any time in the past, and the 0 subscripts refer to the present.

   For a chondrite uniform reservoir (CHUR) the following values are given for the Sm-Nd isotopic system:

$$\left(\frac{^{143}\mathrm{Nd}}{^{144}\mathrm{Nd}}\right)_0 = 0.512638 \qquad \left(\frac{^{147}\mathrm{Sm}}{^{144}\mathrm{Nd}}\right)_0 = 0.1967 \quad \lambda_{147} = 6.54 \times 10^{-12}/\mathrm{yr}$$

   For a uniform mantle reservoir of bulk earth (BE) composition, the following values are given for the Rb-Sr isotopic system:

$$\left(\frac{^{87}\mathrm{Sr}}{^{86}\mathrm{Sr}}\right)_0 = 0.7045 \qquad \left(\frac{^{87}\mathrm{Rb}}{^{86}\mathrm{Sr}}\right)_0 = 0.0816 \quad \lambda_{87} = 1.42 \times 10^{-11}/\mathrm{yr}$$

Plot the evolution curves for $\frac{^{143}Nd}{^{144}Nd}$ and $\frac{^{87}Sr}{^{86}Sr}$ in CHUR and BE, respectively, from 4.5 billion years to the present, and determine what values these isotopic ratios had 2.5 x 10^9 years ago.

b. Using the equations in (a) above, calculate the present day ratios of $\frac{^{143}Nd}{^{144}Nd}$ and $\frac{^{87}Sr}{^{86}Sr}$ in the residual mantle and resulting partial melt assuming that the original mantle had ratios equivalent to CHUR and BE when the melting event took place 2.5 x 10^9 years ago and had the following values after the melting event:

	Residual Mantle	Partial Melt
$^{147}Sm/^{144}Nd$	0.2698	0.1000
$^{87}Rb/^{86}Sr$	0.0208	0.2140

Note that you will first have to convert the ratios of $^{147}Sm/^{144}Nd$ and $^{87}Rb/^{86}Sr$ to their present day ratios using the equation:

$$N = N_0 e^{-\lambda t}$$

where N = the # atoms today and N_0 = # atoms at some time in the past.

Plot the evolution curves for the residual mantle and partial melt for both the Sm-Nd and Rb-Sr systems on the same diagram you constructed in part (a).

2. Calculate the heat production per gram (P) for the isotopes ^{40}K, ^{238}U, ^{235}U, and ^{232}Th from the present to 4.5 billion years ago. In addition, calculate the total heat produced by the decay of these radio-nuclides. Plot the results against time measured backwards from the present. To accomplish these calculations, consider the following:

a) The concentration of an isotope (C) at some time (t) in the past (measured positively from the present) can be expressed as:

$$C = C_o \exp(\lambda t) \qquad (1)$$

where C_o is the concentration of the isotope today and λ refers to the decay constant for the nuclide. Show that equation (1) is equivalent to:

$$C = C_o \exp\left(\frac{t \ln 2}{\tau_{1/2}}\right) \qquad (2)$$

where $\tau_{1/2}$ stands for the half life of the radio-nuclide.

b) The present day heat production (P_o) due to the elements U, Th, and K can be written

$$P_o = C_o^U[P^U + (C_o^{Th}/C_o^U)P^{Th} + (C_o^K/C_o^U)P^K] \qquad (3)$$

where P^U, P^{Th}, and P^K stand for the bulk heat productivity for the elements U, Th, and K, respectively. Given the following data consistent with the mean heat production of the mantle today:

$P_o = 6.187 \times 10^{-12}$ W/kg $\qquad P^U = 9.711 \times 10^{-5}$ W/kg,

$P^{Th} = 2.696 \times 10^{-5}$ W/kg $\qquad P^K = 3.579 \times 10^{-9}$ W/kg,

$C_o^{Th}/C_o^U = 4$ $\qquad C_o^K/C_o^U = 10^4$

calculate the present day concentrations of U, Th, and K in the mantle. Note, your result will be in units of weight percent.

c) Given that the heat production of some isotope, i, of some element, e, is given by

$${}^iP^e = {}^ip^e\ {}^iC^e$$

where ${}^ip^e$ is the heat production rate of the isotope, i, of element, e, and ${}^iC^e$ is the concentration of isotope i, of element e, use the results from above and the information below to calculate the time dependence of the heat production of each isotope at various times in the past.

Sum the contributions of the individual isotopes to find the total heat production (P) at any time in the past. Plot a graph showing the total heat production, and the heat production of each isotope as a function of time from 4.5 billion years ago to the present. Comment on relative importance of the various isotopes as heat producers at present and during the Earth's past.

Isotope	Heat Production Rate (Watts/Kg)	Abundance	Half Life (yr)
${}^{238}U$	9.38×10^{-5}	${}^{238}C_o^U = 0.9927\ C_o^U$	4.47×10^9
${}^{235}U$	5.69×10^{-4}	${}^{235}C_o^U = 0.0072\ C_o^U$	7.04×10^8
${}^{232}Th$	2.70×10^{-5}	${}^{232}C_o^{Th} = C_o^{Th}$	1.40×10^{10}
${}^{40}K$	2.80×10^{-11}	${}^{40}C_o^K = 1.28 \times 10^{-4} C_o^K$	1.25×10^9

3. Determine the present day rates of heat production for the following rocks. Calculate the present day concentrations of each isotope, using the table above, multiply by the heat production rate for each isotope, again using the table above, and then sum the heat production rates for each isotope to determine the total heat production rate for each rock.

	U (ppm)	Th (ppm)	K (wt. %)
Reference "undepleted" mantle	0.026	0.103	0.026
"Depleted" peridotites	0.012	0.035	0.004
Tholeiitic basalt	0.1	0.35	0.2
Granite	4	17	3.2
Chondritic meteorites	0.013	0.04	0.078

Note that 1 wt. percent equals 10,000 ppm.

4. The differential equation governing the steady state heat flow is:

$$\nabla^2 T = \frac{\partial^2 T}{\partial z} = -\frac{\varepsilon}{K}$$

We wish to solve this equation and determine geothermal gradients (temperature distribution) in the crust using various values of crustal thickness, heat production, thermal conductivity, and input of heat from the mantle.

a. Derive an equation for the variation of temperature with depth, z, assuming the following boundary conditions:

$T = 0$ at $z = 0$
and $q = -q_m$ at $z = d$

where q_m refers to a constant upward heat flow from the mantle.
Note, use the second boundary condition to evaluate the first integration constant, since

$$\frac{\partial T}{\partial z} = \frac{q_m}{K} \qquad \text{at } z = d$$

and use the first boundary condition to evaluate the second integration constant.

b. Now use the equation derived in (a)

hint, it looks like this → $$T = \frac{\varepsilon z^2}{2K} + \left(\frac{q_m + \varepsilon d}{K}\right)z$$

to plot the temperature distribution in the crust with a thickness of 50 km, with

$K = 2.5 W/(m^oC)$
$\varepsilon = 1 \times 10^{-6} W/m^3$
$q_m = 0.021 W/m^2$

c. Plot the same curves with the following changes

(i) Reduce the thermal conductivity to 1.7 $W/(m^oC)$

(ii) Increase the heat production to 2×10^{-6} W/m^3

(iii) Increase the basal heat flow (q_m to $0.042 W/m^2$)

d. Calculate the shallow geothermal gradient (in units of $^oC/km$) for each of the conditions in b and c, above, and comment on the effects on the shallow gradient of reducing the thermal conductivity in the crust, increasing the heat production in the crust, and increasing the basal heat flow into the crust.

5. Given that the Laplacian $\nabla^2 T$ in spherical coordinates can be expressed as:

$$\nabla^2 T = \left(\frac{1}{r^2}\right)\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) = \frac{d^2 T}{d r^2} \quad (5)$$

a) recalling that for the steady state, uniform heat source case, $\nabla^2 T = -\varepsilon / K$, show that the temperature distribution within the sphere is give as

$$T = T_o + \varepsilon (a^2 - r^2)/6K \quad (6)$$

where a is the radius of the sphere. Note that the temperature at the center of the sphere (r = 0) must be finite, and the temperature at the surface of the sphere (r = a) must be T_o.

b) What would be temperature at the center of the Earth for this model assuming a = 6.371×10^6 m, $q_s = 80 \times 10^{-3}$ W/m^2, and $K = 4 W/(m\ ^oC)$?

Hint: to solve this problem you must first find ε. Recall that $q_s = -KdT/dr$

6. The Rayleigh number (R_a) is a dimensionless number that expresses the ratio of buoyant forces (those enhancing the possibility of convection) to viscous forces (those impeding the possibility of convection) in any material. One way of expressing the Rayleigh number is:

$$R_a = \frac{\alpha\beta g d^4}{k\nu}$$

where α is the coefficient of thermal expansion, g the acceleration due to gravity, β is the temperature gradient in excess of the adiabatic gradient, d is the thickness of the layer, k is the thermal diffusivity, and ν is the kinematic viscosity (viscosity/density).

(a) Use the following values to calculate R_a in the upper mantle, lower mantle, and whole mantle.

$\alpha = 2 \times 10^{-5}$ /°C $k = 10^{-6} m^2/s$

For upper mantle
$\nu = 10^{17} m^2/s$ d = 700 km $\beta = 10^{-3}$ °C/m

For lower mantle
$\nu = 10^{16} m^2/s$ d = 2000 km $\beta = 10^{-4}$ °C/m

For whole mantle
$\nu = 5 \times 10^{16} m^2/s$ d = 2700 km $\beta = 10^{-4}$ °C/m

(b) Considering that the critical Rayleigh number for convection to occur under most conditions is about 10^3, what do you conclude about the possibility that the mantle is transferring its heat to the crust by convection?

(c) For the whole mantle, calculate the minimum temperature gradient in excess of the adiabatic gradient that is necessary for the mantle to convect. Assume that the critical Rayleigh number is 2000. Express your answer in units of °C/km.