Why is $\pi$ the same in $C = 2\pi r$ and $A = \pi r^2$?

Approximating $\pi$ Using Areas of Polygons—the Method of Archimedes

Sandwiching $\pi$

A circle of radius $r$ has area $\pi r^2$, so the area of a circle of radius 1 is $\pi$. Therefore $\pi$ is sandwiched between the areas of the regular $n$-sided polygons inscribed and circumscribed around the circle. \texttt{PiPolygons[n]} draws the three figures and prints the corresponding inequality for the areas.

\begin{Verbatim}
PiPolygons[5];
\end{Verbatim}

\begin{center}
\includegraphics[width=\textwidth]{circle_polygons.png}
\end{center}

\begin{Verbatim}
2.37764 \leq \pi \leq 3.63271
\end{Verbatim}
Finding the Area of the Inscribed Polygon

Break the inscribed polygon into \( n \) triangles of height \( \cos \left( \frac{a}{2} \right) \) and base \( 2 \sin \left( \frac{a}{2} \right) \), where \( a \) is the angle \( 2 \pi / n \).

The area is \( n \times 1/2 \times \text{base} \times \text{height} = n/2 (2 \sin(\alpha)/2 \cos(\alpha)/2) = \frac{n}{2} \sin(a) \).

Finding the Area of the Circumscribed Polygon

For the circumscribed polygon, the \( n \) triangles each have height 1 and base \( 2 \tan(\alpha)/2 \). The area is \( n \times 1/2 \times 2 \times \tan(\alpha)/2 = n \tan(\alpha)/2 \).

This shows a triangle from an inscribed and circumscribed 7-sided polygon.

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PiTriangles[7];
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Vieta's Formula for \( \pi \)

We continue the discussion on approximating \( \pi \) by calculating the areas of a circle using inscribed and circumscribed regular polygons. We illustrate Vieta's formula, developed in 1593, the oldest exact result derived for \( \pi \).

The Formula

Vieta's formula expresses \( \frac{\pi}{2} \) as an infinite product of nested square roots.
\[ \frac{2}{\pi} = \left( \frac{1}{2} \sqrt{2} \right) \left( \frac{1}{2} \sqrt{2 + \sqrt{2}} \right) \left( \frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2}}} \right) \ldots \]

Quality of Approximation

The graph in the middle is the approximation from Vieta's formula, which calculates the area of a regular polygon of \(2^{n+1}\) sides. In contrast, the upper and lower graphs show the estimates to \(\pi\) based on the areas of the \(n\)-sided regular polygons circumscribed and inscribed around a circle of radius 1.

\[ \text{PiSqueeze[100]}; \]

The second column shows the number of sides of the regular polygons, while the third shows the approach to \(\pi\). The numbers in the fourth column are the differences between \(\pi\) and the partial products of the infinite product.
Why is It True?

### Computing π with Series

### Computing π by Iterative Processes

### Implementation